$$h_{[\omega]}\colon \pi_n(X,A,\omega(1)) \to \pi_n(X,A,\omega(0))$$

In this way, $\pi_1(A,x_0)$ *acts as a group of operators on the left on* $\pi_n(X,A,x_0)$, *and if A is path connected and* $x_0, x_1 \in A$, *then* $\pi_n(X,A,x_0)$ *and* $\pi_n(X,A,x_1)$ *are isomorphic by an isomorphism determined up to the action of* $\pi_1(A,x_0)$. ∎

If ω is a path in A, it follows from lemma 1a that there is a commutative square for $n > 1$,

$$\begin{array}{ccc} \pi_n(X,A,\omega(1)) & \xrightarrow{\partial} & \pi_{n-1}(A,\omega(1)) \\ {\scriptstyle h_{[\omega]}}\downarrow & & \downarrow{\scriptstyle h_{[\omega]}} \\ \pi_n(X,A,\omega(0)) & \xrightarrow{\partial} & \pi_{n-1}(A,\omega(0)) \end{array}$$

Thus there is also a covariant functor from the fundamental groupoid of A to the category of exact sequences which assigns to $x \in A$ the homotopy sequence of (X,A,x).

A pair (X,A) with A path connected is said to be *n-simple* (for $n \geq 1$) if $\pi_1(A,x_0)$ acts trivially on $\pi_n(X,A,x_0)$ for some (and hence all) base points $x_0 \in A$. If A is simply connected, (X,A) is n-simple for every $n \geq 1$.

11 THEOREM *Let* (X,A) *be a pair of H spaces with A path connected. Then* (X,A) *is n-simple for all* $n \geq 1$.

PROOF This is immediate from theorem 5. ∎

If (X,A) is n-simple and $x_0, x_1 \in A$, then $\pi_n(X,A,x_0)$ and $\pi_n(X,A,x_1)$ are canonically isomorphic. Therefore any map $\alpha\colon (E^n,S^{n-1}) \to (X,A)$ determines a unique element of $\pi_n(X,A,x_0)$ (whether or not α maps the base point $p_0 \in S^{n-1}$ to x^0), and $\pi_n(X,A,x_0)$ is in one-to-one correspondence with the free homotopy classes $[E^n,S^{n-1}; X,A]$. If (X,A) is n-simple, we shall frequently omit the base point and write $\pi_n(X,A)$.

The action of $\pi_1(A,x_0)$ on $\pi_2(X,A,x_0)$ is closely related to conjugation, as shown by the next result.

12 THEOREM *If* $a, b \in \pi_2(X,A,x_0)$, *then*

$$aba^{-1} = h_{\partial a}(b)$$

PROOF Let $X' = P(X,x_0)$ and let $p\colon X' \to X$ be the path fibration. Let $A' = p^{-1}(A)$ and let $x_0' \in A'$ be the constant path at x_0. By theorem 7.2.8, there is an isomorphism

$$p_{\#}\colon \pi_2(X',A',x_0') \approx \pi_2(X,A,x_0)$$

Let $a' = p_{\#}{}^{-1}(a)$ and $b' = p_{\#}{}^{-1}(b)$ and observe that, by lemma 1b,

$$h_{\partial a}(b) = p_{\#}(h_{\partial a'}(b'))$$

Hence it suffices to prove that $a'b'a'^{-1} = h_{\partial a'}(b')$. Because X' is contractible, it follows from the exactness of the homotopy sequence of (X',A',x_0') that

$$\partial\colon \pi_2(X',A',x_0') \approx \pi_1(A',x_0')$$

So to complete the proof we need only prove that

$$\partial(a'b'a'^{-1}) = \partial(h_{\partial a'}(b'))$$

The left-hand side equals $(\partial a')(\partial b')(\partial a')^{-1}$, and because ∂ commutes with $h_{\partial a'}$, the right-hand side equals $h_{\partial a'}(\partial b')$. The result now follows from the fact that the action of $\pi_1(A',x_0')$ on itself given by h is the same as conjugation. ∎

This again implies that $\pi_2(X,x_0) \approx \pi_2(X,\{x_0\},x_0)$ is abelian. Together with the exactness of the homotopy sequence, it yields the next result.

13 COROLLARY *The inclusion map j: $(X,x_0) \subset (X,A)$ induces a homomorphism*

$$j_\#\colon \pi_2(X,x_0) \to \pi_2(X,A,x_0)$$

whose image is in the center of $\pi_2(X,A,x_0)$. ∎

The following result is a generalization of theorem 1.8.7 to the higher relative homotopy groups.

14 THEOREM *Let f: $(X,A,x_0) \to (Y,B,y_0)$ and g: $(X,A,x_0) \to (Y,B,y_1)$ be freely homotopic. Then there is a path ω in B from y_0 to y_1 such that*

$$f_\# = h_{[\omega]} \circ g_\#\colon \pi_n(X,A,x_0) \to \pi_n(Y,B,y_0) \qquad n \geq 2$$

PROOF Let F: $(X,A) \times I \to (Y,B)$ be a homotopy from $f \mid (X,A)$ to $g \mid (X,A)$ and let $\omega(t) = F(x_0,t)$. Then ω is a path in B from y_0 to y_1, and if α: $(I^n,\dot{I}^n,p_0) \to (X,A,x_0)$ represents an element of $\pi_n(X,A,x_0)$, then the composite

$$(I^n,\dot{I}^n) \times I \xrightarrow{\alpha \times 1} (X,A) \times I \xrightarrow{F} (Y,B)$$

is an ω-homotopy from $f \circ \alpha$ to $g \circ \alpha$. Therefore

$$f_\#[\alpha] = [f \circ \alpha] = h_{[\omega]}([g \circ \alpha]) = (h_{[\omega]} \circ g_\#)[\alpha] \quad ∎$$

This yields the following analogue of theorem 1.8.8.

15 COROLLARY *Let f: $(X,A) \to (Y,B)$ be a homotopy equivalence. For any $x \in A$, f induces isomorphisms*

$$f_\#\colon \pi_n(X,A,x) \approx \pi_n(Y,B,f(x))$$

PROOF Let g: $(Y,B) \to (X,A)$ be a homotopy inverse of f. By theorem 14, there are paths ω in A from $gf(x)$ to x and ω' in B from $fgf(x)$ to $f(x)$ such that the following diagram is commutative

$$\begin{tikzcd}
\pi_n(X,A,x) \arrow[r, "h_{[\omega]}"] \arrow[d, "f_\#"'] & \pi_n(X,A,gf(x)) \arrow[d, "f_\#"] \\
\pi_n(Y,B,f(x)) \arrow[r, "h_{[\omega']}"] \arrow[ur, "g_\#"] & \pi_n(Y,B,fgf(x))
\end{tikzcd}$$

Since the maps $h_{[\omega]}$ and $h_{[\omega']}$ are isomorphisms, all the maps in the diagram are isomorphisms. ∎

4 THE HUREWICZ HOMOMORPHISM

There are no algorithms for computing the absolute or relative homotopy groups of a topological space (even when the space is given with a triangulation). One of the few main tools available for the general study of homotopy groups is their comparison with the corresponding integral singular homology groups. Such a comparison is effected by means of a canonical homomorphism from homotopy groups to homology groups. The definition and functorial properties of this homomorphism are our concern in this section. A theorem asserting that in the lowest nontrivial dimension for the homotopy group this homomorphism is an isomorphism will be established in the next section.

We shall be working with the integral singular homology theory throughout this section. Let $n \geq 1$ and recall that $H_q(I^n,\dot{I}^n) = 0$ for $q \neq n$ and $H_n(I^n,\dot{I}^n)$ is infinite cyclic. To consider relations among the homology groups of certain pairs in I^n, for $n \geq 1$ we define

$$\begin{aligned}
I_1{}^n &= \{(t_1, \ldots ,t_n) \in I^n \mid t_n \leq \tfrac{1}{2}\}\\
\dot{I}_1{}^n &= (I_1{}^n \cap \dot{I}^n) \cup \{(t_1, \ldots ,t_n) \in I^n \mid t_n = \tfrac{1}{2}\}\\
I_2{}^n &= \{(t_1, \ldots ,t_n) \in I^n \mid t_n \geq \tfrac{1}{2}\}\\
\dot{I}_2{}^n &= (I_2{}^n \cap \dot{I}^n) \cup \{(t_1, \ldots ,t_n) \in I^n \mid t_n = \tfrac{1}{2}\}
\end{aligned}$$

Then $I_1{}^n \cup I_2{}^n = I^n$ and $(I_1{}^n \cup \dot{I}_2{}^n) \cap (\dot{I}_1{}^n \cup I_2{}^n) = \dot{I}_1{}^n \cup \dot{I}_2{}^n$. By the exactness of the Mayer-Vietoris sequence of the excisive couple $\{I_1{}^n \cup \dot{I}_2{}^n, \dot{I}_1{}^n \cup I_2{}^n\}$, we have

$$H_q(I_1{}^n \cup \dot{I}_2{}^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n) \oplus H_q(\dot{I}_1{}^n \cup I_2{}^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n) \approx H_q(I^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n)$$

By excision, we also have isomorphisms

$$\begin{aligned}
H_q(I_1{}^n,\dot{I}_1{}^n) &\approx H_q(I_1{}^n \cup \dot{I}_2{}^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n)\\
H_q(I_2{}^n,\dot{I}_2{}^n) &\approx H_q(\dot{I}_1{}^n \cup I_2{}^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n)
\end{aligned}$$

Combining these, we see that if we let i_1: $(I_1{}^n,\dot{I}_1{}^n) \subset (I^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n)$ and we let i_2: $(I_2{}^n,\dot{I}_2{}^n) \subset (I^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n)$, then we have the following result.

1 LEMMA *The inclusion maps i_1 and i_2 define a direct-sum representation*

$$i_{1*} \oplus i_{2*} : H_q(I_1{}^n,\dot{I}_1{}^n) \oplus H_q(I_2{}^n,\dot{I}_2{}^n) \approx H_q(I^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n) \quad \blacksquare$$

Let ν_1: $(I^n,\dot{I}^n) \to (I_1{}^n,\dot{I}_1{}^n)$ be defined by $\nu_1(t_1, \ldots ,t_n) = (t_1, \ldots ,t_{n-1},t_n/2)$ and define ν_2: $(I^n,\dot{I}^n) \to (I_2{}^n,\dot{I}_2{}^n)$ by $\nu_2(t_1, \ldots ,t_n) = (t_1, \ldots ,t_{n-1}, (t_n + 1)/2)$. Let i: $(I^n,\dot{I}^n) \subset (I^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n)$.

2 COROLLARY *For any $z \in H_n(I^n,\dot{I}^n)$*

$$i_* z = i_{1*}\nu_{1*} z + i_{2*}\nu_{2*} z$$

PROOF Let j_1: $(I^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n) \subset (I^n, I_1{}^n \cup \dot{I}_2{}^n)$ and j_2: $(I^n, \dot{I}_1{}^n \cup \dot{I}_2{}^n) \subset (I^n, \dot{I}_1{}^n \cup I_2{}^n)$. Then $j_{1*} i_{1*} = 0$ and $j_{1*} i_{2*}$ is an isomorphism of $H_q(I_2{}^n,\dot{I}_2{}^n)$

onto $H_q(I^n, I_1{}^n \cup \dot{I}_2{}^n)$ (induced by the inclusion map, which is an excision). Similarly, $j_{2*}i_{2*} = 0$ and $j_{2*}i_{1*}$ is an isomorphism of $H_q(I_1{}^n,\dot{I}_1{}^n)$ onto $H_q(I^n, \dot{I}_1{}^n \cup I_2{}^n)$. It follows from lemma 1 that

$$\ker j_{1*} \cap \ker j_{2*} = 0$$

Therefore, to prove the corollary it suffices to prove that

$$i_* z - i_{1*}\nu_{1*}z - i_{2*}\nu_{2*}z$$

is in the kernel of j_{1*} and in the kernel of j_{2*}.

We first prove that $j_{1*}(i_* z - i_{1*}\nu_{1*}z - i_{2*}\nu_{2*}z) = 0$. Because $j_{1*}i_{1*} = 0$, we must show that $j_{1*}i_* z = j_{1*}i_{2*}\nu_{2*}z$. Clearly $j_1 i$ is the inclusion map $(I^n,\dot{I}^n) \subset (I^n, I_1{}^n \cup \dot{I}_2{}^n)$ and $j_1 i_2 \nu_2$ is the map $f\colon (I^n,\dot{I}^n) \to (I^n, I_1{}^n \cup \dot{I}_2{}^n)$ defined by $f(t_1, \ldots ,t_n) = (t_1, \ldots ,t_{n-1}, (t_n + 1)/2)$. A homotopy H from $j_1 i$ to f is defined by

$$H((t_1, \ldots ,t_n), t) = (t_1, \ldots ,t_{n-1}, (t_n + t)/(1 + t))$$

Therefore $j_{1*}i_* = f_* = j_{1*}i_{2*}\nu_{2*}$. A similar argument shows that

$$j_{2*}(i_* z - i_{1*}\nu_{1*}z - i_{2*}\nu_{2*}z) = 0 \quad \blacksquare$$

For $n \geq 1$ the subset $I \times \dot{I}^{n-1} \cup 0 \times I^{n-1} \subset \dot{I}^n$ is contractible. Therefore $H_q(I^n, I \times \dot{I}^{n-1} \cup 0 \times I^{n-1}) = 0$ for all q. By exactness of the homology sequence of the triple $(I^n, \dot{I}^n, I \times \dot{I}^{n-1} \cup 0 \times I^{n-1})$, it follows that the map

$$\partial\colon H_q(I^n,\dot{I}^n) \to H_{q-1}(\dot{I}^n, I \times \dot{I}^{n-1} \cup 0 \times I^{n-1})$$

is an isomorphism for all q. For $n \geq 2$ let

$$j\colon (I^{n-1},\dot{I}^{n-1}) \to (\dot{I}^n, I \times \dot{I}^{n-1} \cup 0 \times I^{n-1})$$

be defined by $j(t_1, \ldots ,t_{n-1}) = (1, t_1, \ldots ,t_{n-1})$. Then j is the composite of a homeomorphism from $(I^{n-1},\dot{I}^{n-1})$ to $(1 \times I^{n-1}, 1 \times \dot{I}^{n-1})$ and the excision map

$$(1 \times I^{n-1}, 1 \times \dot{I}^{n-1}) \subset (\dot{I}^n, I \times \dot{I}^{n-1} \cup 0 \times I^{n-1})$$

Therefore the homomorphism

$$j_*\colon H_q(I^{n-1},\dot{I}^{n-1}) \to H_q(\dot{I}^n, I \times \dot{I}^{n-1} \cup 0 \times I^{n-1})$$

is an isomorphism for all q.

We define canonical generators $Z_n \in H_n(I^n,\dot{I}^n)$ for $n \geq 1$ by induction on n as follows:

(*a*) $Z_1 \in H_1(I,\dot{I})$ is the unique element with $\partial Z_1 = (1) - (0)$ in $H_0(\dot{I})$.
(*b*) For $n \geq 2$, $Z_n \in H_n(I^n,\dot{I}^n)$ is the unique element such that $\partial Z_n = j_* Z_{n-1}$ in $H_{n-1}(\dot{I}^n, I \times \dot{I}^{n-1} \cup 0 \times I^{n-1})$.

Given a map $\alpha\colon (I^n,\dot{I}^n) \to (X,A)$, then $\alpha_* Z_n \in H_n(X,A)$. If $\alpha \simeq \beta$, then $\alpha_* Z_n = \beta_* Z_n$. Therefore there is for $n \geq 1$ a well-defined map

$$\varphi\colon \pi_n(X,A,x_0) \to H_n(X,A)$$

such that $\varphi[\alpha] = \alpha_* Z_n$, where $\alpha: (I^n,\dot I^n) \to (X,A)$ maps z_0 to x_0 and represents an element of $\pi_n(X,A,x_0)$. By identifying $\pi_n(X,x_0)$ with $\pi_n(X,\{x_0\},x_0)$, we also have a map $\varphi: \pi_n(X,x_0) \to H_n(X,x_0)$. Some of the basic properties of φ are summarized in the next result.

3 THEOREM *If $n \geq 2$ or if $n = 1$ and $A = \{x_0\}$, the map φ is a homomorphism. It has the following functorial properties:*

(*a*) *For $n \geq 2$ commutativity holds in the square*

$$\begin{array}{ccc} \pi_n(X,A,x_0) & \xrightarrow{\partial} & \pi_{n-1}(A,x_0) \\ \varphi\downarrow & & \downarrow\varphi \\ H_n(X,A) & \xrightarrow{\partial} & H_{n-1}(A,x_0) \end{array}$$

(*b*) *Given $f: (X,A,x_0) \to (Y,B,y_0)$, commutativity holds in the square*

$$\begin{array}{ccc} \pi_n(X,A,x_0) & \xrightarrow{f_\#} & \pi_n(Y,B,y_0) \\ \varphi\downarrow & & \downarrow\varphi \\ H_n(X,A) & \xrightarrow{f_*} & H_n(Y,B) \end{array}$$

PROOF Let $\alpha_1, \alpha_2: (I^n,\dot I^n) \to (X,A)$ be such that

$$\alpha_1(t_1, \ldots ,t_{n-1}, 1) = \alpha_2(t_1, \ldots ,t_{n-1}, 0)$$

[any two maps of $(I^n,\dot I^n)$ to (X,A) are homotopic to such maps if $n \geq 2$ or if $n = 1$ and $A = \{x_0\}$]. Then $\alpha_1 * \alpha_2 = \beta \circ i$, where $i: (I^n,\dot I^n) \subset (I^n, \dot I_1{}^n \cup \dot I_2{}^n)$ and $\beta: (I^n, \dot I_1{}^n \cup \dot I_2{}^n) \to (X,A)$ is defined by

$$\beta(t_1, \ldots ,t_n) = \begin{cases} \alpha_1(t_1, \ldots ,t_{n-1}, 2t_n) & t_n \leq \tfrac{1}{2} \\ \alpha_2(t_1, \ldots ,t_{n-1}, 2t_n - 1) & t_n \geq \tfrac{1}{2} \end{cases}$$

Then $\varphi[\alpha_1 * \alpha_2] = \beta_* i_* Z_n = \beta_* (i_{1*}\nu_{1*}Z_n + i_{2*}\nu_{2*}Z_n)$, the last equality by corollary 2. Since $\beta i_1\nu_1 = \alpha_1$ and $\beta i_2 \nu_2 = \alpha_2$, we see that

$$\varphi[\alpha_1 * \alpha_2] = \alpha_{1*} Z_n + \alpha_{2*} Z_n = \varphi[\alpha_1] + \varphi[\alpha_2]$$

which shows that φ is a homomorphism whenever $\pi_n(X,A,x_0)$ is a group.

To prove (*a*), let $\alpha: (I^n,\dot I^n) \to (X,A)$ represent an element of $\pi_n(X,A)$ for $n \geq 2$ and suppose that $\alpha(I \times \dot I^{n-1} \cup 0 \times I^{n-1}) = x_0$. Then $\partial[\alpha] = [\alpha']$, where $\alpha': (I^{n-1},\dot I^{n-1}) \to (A,x_0)$ is defined by $\alpha' = (\alpha \mid (\dot I^n, I \times \dot I^{n-1} \cup 0 \times I^{n-1})) \circ j$. Then

$$\begin{aligned} \varphi\partial[\alpha] &= \alpha'_* Z_{n-1} = (\alpha|(\dot I^n, I\times \dot I^{n-1} \cup 0 \times I^{n-1}))_* j_* Z_{n-1} \\ &= (\alpha \mid (\dot I^n, I \times \dot I^{n-1} \cup 0 \times I^{n-1}))_* \partial Z_n \\ &= \partial \alpha_* Z_n = \partial\varphi[\alpha] \end{aligned}$$

Finally, (*b*) follows from the fact that $(f\alpha)_* = f_* \alpha_*$. ■

The map φ is called the *Hurewicz homomorphism*. The next result follows from theorem 3.

4 COROLLARY *The Hurewicz homomorphism maps the homotopy sequence of (X,A,x_0) into the homology sequence of (X,A,x_0).* ∎

Our next objective is to show that the Hurewicz homomorphism commutes with the actions of the appropriate fundamental group on the homotopy set. We consider the relative case first.

5 LEMMA *Let $[\alpha] \in \pi_n(X,A,x_0)$ for $n \geq 2$ and let $[\omega] \in \pi_1(A,x_0)$. Then*

$$\varphi(h_{[\omega]}[\alpha]) = \varphi[\alpha]$$

PROOF Let $[\alpha]$ be represented by α: $(I^n,\dot{I}^n) \to (X,A)$ and let $h_{[\omega]}[\alpha]$ be represented by α': $(I^n,\dot{I}^n) \to (X,A)$. Then α and α' are freely homotopic [that is, α and α' are homotopic as maps of $(I^n,\dot{I}^n)$ to (X,A)]. Therefore

$$\varphi[\alpha] = \alpha_* Z_n = \alpha'_* Z_n = \varphi[\alpha'] = \varphi(h_{[\omega]}[\alpha]) \quad ∎$$

Next we prove the corresponding result for the absolute case.

6 LEMMA *Let $[\alpha] \in \pi_n(X,x_0)$ and $[\omega] \in \pi_1(X,x_0)$. Then*

$$\varphi(h_{[\omega]}[\alpha]) = \varphi[\alpha]$$

PROOF Let Y be the space obtained from I^n by collapsing $\dot{I}^n$ to a single point, this point to be the base point of Y, denoted by y_0. The collapsing map g: $(I^n,\dot{I}^n) \to (Y,y_0)$ induces a one-to-one correspondence between $[Y,y_0;\ X,x_0]$ and $[I^n,\dot{I}^n;\ X,x_0]$. Therefore $\pi_n(X,x_0)$ can be identified with $[Y,y_0;\ X,x_0]$. Furthermore, $g_* : H_n(I^n,\dot{I}^n) \approx H_n(Y,y_0)$, and we let $g_* Z_n = Z'_n \in H_n(Y,y_0)$. In these terms, if an element of $\pi_n(X,x_0)$ is represented by α: $(Y,y_0) \to (X,x_0)$, then $\varphi[\alpha] = \alpha_* Z'_n$. Let $h_{[\omega]}[\alpha]$ be represented by α': $(Y,y_0) \to (X,x_0)$. Then α and α' are homotopic as maps of Y to X. Therefore, if $Z''_n \in H_n(Y)$ is the unique element such that $i'_* Z''_n = Z'_n$ [where i': $Y \subset (Y,y_0)$], then

$$(\alpha \mid Y)_* Z''_n = (\alpha' \mid Y)_* Z''_n$$

Let j': $X \subset (X,x_0)$. Then

$$\varphi[\alpha] = \alpha_* Z'_n = \alpha_* i'_* Z''_n = j'_* (\alpha \mid Y)_* Z'_n$$

Similarly, $\varphi[\alpha'] = j'_* (\alpha' \mid Y)_* Z''_n$, and

$$\varphi[\alpha] = \varphi[\alpha'] = \varphi(h_{[\omega]}[\alpha]) \quad ∎$$

We define $\pi'_n(X,A,x_0)$ for $n \geq 2$ to be the quotient group of $\pi_n(X,A,x_0)$ by the normal subgroup G generated by

$$\{(h_{[\omega]}[\alpha])[\alpha]^{-1} \mid [\alpha] \in \pi_n(X,A,x_0),\ [\omega] \in \pi_1(A,x_0)\}$$

By lemma 5, φ maps G to 0 and there is a homomorphism

$$\varphi': \pi'_n(X,A,x_0) \to H_n(X,A)$$

whose composite with the canonical map η: $\pi_n(X,A,x_0) \to \pi'_n(X,A,x_0)$ is φ. Note that, by theorem 7.3.12, $\pi'_n(X,A,x_0)$ is abelian for all $n \geq 2$.

Similarly, we define $\pi'_n(X,x_0)$ for $n \geq 1$ to be the quotient group of

$\pi_n(X,x_0)$ by the normal subgroup H generated by

$$\{(h_{[\omega]}[\alpha])[\alpha]^{-1} \mid [\alpha] \in \pi_n(X,x_0),\ [\omega] \in \pi_1(X,x_0)\}$$

By lemma 6, φ maps H to 0, and there is a homomorphism

$$\varphi'\colon \pi'_n(X,x_0) \to H_n(X,x_0)$$

whose composite with the canonical map $\eta\colon \pi_n(X,x_0) \to \pi'_n(X,x_0)$ is φ. Note that $\pi'_1(X,x_0)$ is the quotient group of $\pi_1(X,x_0)$ by its commutator subgroup. In particular, $\pi'_n(X,x_0)$ is abelian for all $n \geq 1$.

Because the groups $\pi'_n(X,A,x_0)$ and $\pi'_n(X,x_0)$ are abelian, we shall find them easier to compare with the homology groups (which are abelian) than the homotopy groups themselves. For the comparison it will be convenient to replace the triple $(I^n,\dot{I}^n,z_0)$, which is the antecedent triple used to define $\pi_n(X,A,x_0)$, by the homeomorphic triple $(\Delta^n,\dot{\Delta}^n,v_0)$, where Δ^n is the standard n-simplex used in Sec. 4.1 to define the singular complex (vertices of Δ^n will be denoted by $v_0, v_1, \ldots, v_n$). To achieve this replacement we need only choose a homeomorphism of $(\Delta^n,\dot{\Delta}^n,v_0)$ onto $(I^n,\dot{I}^n,z_0)$. Any homeomorphism $h\colon (\Delta^n,\dot{\Delta}^n) \to (I^n,\dot{I}^n)$ will induce an isomorphism

$$h_*\colon H_n(\Delta^n,\dot{\Delta}^n) \approx H_n(I^n,\dot{I}^n)$$

The identity map $\xi_n\colon \Delta^n \subset \Delta^n$ is a singular simplex which is a cycle modulo $\dot{\Delta}^n$ and whose homology class $\{\xi_n\}$ is a generator of the infinite cyclic group $H_n(\Delta^n,\dot{\Delta}^n)$. Since Z_n is a generator of $H_n(I^n,\dot{I}^n)$ and h_* is an isomorphism, either $h_*\{\xi_n\} = Z_n$ or $h_*\{\xi_n\} = -Z_n$. We want to choose h so that the former holds. If $n = 1$, the choice of Z_1 is such that the simplicial homeomorphism $h\colon \Delta^1 \to I$ with $h(v_0) = 0$ and $h(v_1) = 1$ will have the desired property (that is, $h_*\{\xi_1\} = Z_1$). If $n > 1$, we choose an arbitrary homeomorphism $h\colon (\Delta^n,\dot{\Delta}^n) \to (I^n,\dot{I}^n)$ such that $h(v_0) = z_0$. If $h_*\{\xi_n\} = -Z_n$, we replace h by $h\lambda$, where λ is a simplicial homeomorphism of Δ^n to itself such that $\lambda(v_0) = v_0$ and $\lambda_*\{\xi_n\} = -\{\xi_n\}$ (for example, λ is the simplicial map which interchanges v_1 and v_2 and leaves all other vertices of Δ^n fixed). Therefore, in any event, we can find a homeomorphism $h\colon (\Delta^n,\dot{\Delta}^n,v_0) \to (I^n,\dot{I}^n,z_0)$ such that $h_*\{\xi_n\} = Z_n$. Using such a homeomorphism to represent elements of $\pi_n(X,A,x_0)$ by maps $\alpha\colon (\Delta^n,\dot{\Delta}^n) \to (X,A)$ such that $\alpha(v_0) = x_0$, we see that $\varphi[\alpha] = \alpha_*\{\xi_n\} = \{\alpha\}$, the latter being the homology class in (X,A) of the singular simplex α.

For any pair (X,A) with base point $x_0 \in A$ and any $n \geq 0$, let $\Delta(X,A,x_0)^n$ be the subcomplex of $\Delta(X)$ generated by singular simplexes $\sigma\colon \Delta^q \to X$ having the property that σ maps each vertex of Δ^q to x_0 and maps the n-dimensional skeleton $(\Delta^q)^n$ of Δ^q into A. Then $\Delta(X,A,x_0)^{n+1} \subset \Delta(X,A,x_0)^n$, and these two chain complexes agree in degrees $\leq n$. Thus we have a decreasing sequence of subcomplexes $\Delta(X,A,x_0)^n$ (where $n \geq 0$) of $\Delta(X)$ whose intersection is contained in $\Delta(A)$. If X is path connected and (X,A) is n-connected for some $n \geq 0$, we shall see that the inclusion map $\Delta(X,A,x_0)^n \subset \Delta(X)$ is a chain equivalence. The following lemma will be used for this purpose.

7 LEMMA *Let C be a subcomplex of the free chain complex $\Delta(X)$ such that C is generated by the singular simplexes of X in it. Assume that to every singular simplex σ: $\Delta^q \to X$ there is assigned a map $P(\sigma)$: $\Delta^q \times I \to X$ such that*

(*a*) *$P(\sigma)(z,0) = \sigma(z)$ for $z \in \Delta^q$.*
(*b*) *Define $\bar\sigma$: $\Delta^q \to X$ by $\bar\sigma(z) = P(\sigma)(z,1)$. Then $\bar\sigma$ is a singular simplex in C, and if σ is in C, $\bar\sigma = \sigma$.*
(*c*) *If e_q^i: $\Delta^{q-1} \to \Delta^q$ omits the ith vertex, then $P(\sigma) \circ (e_q^i \times 1) = P(\sigma^{(i)})$.*

Then the inclusion map $C \subset \Delta(X)$ is a chain equivalence.

PROOF Let j: $C \subset \Delta(X)$ be the inclusion chain map and let τ: $\Delta(X) \to C$ be the chain map defined by $\tau(\sigma) = \bar\sigma$ [(*c*) implies that τ is a chain map]. By (*b*), $\tau \circ j = 1_C$, hence to complete the proof we need only verify that $j \circ \tau \simeq 1_{\Delta(X)}$.

For any space Y let h_0, h_1: $Y \to Y \times I$ be the maps $h_0(y) = (y,0)$ and $h_1(y) = (y,1)$. In the proof of theorem 4.4.3 it was shown (by the method of acyclic models) that there exists a natural chain homotopy D: $\Delta(Y) \to \Delta(Y \times I)$ from $\Delta(h_0)$ to $\Delta(h_1)$. Define a chain homotopy

$$D'\colon \Delta(X) \to \Delta(X)$$

by $D'(\sigma) = \Delta(P(\sigma))(D(\xi_q))$, where σ: $\Delta^q \to X$ and ξ_q: $\Delta^q \subset \Delta^q$. By (*c*), D' is a chain homotopy, and by (*a*) and the definition of $\bar\sigma$, D' is a chain homotopy from $1_{\Delta(X)}$ to $j \circ \tau$. ■

8 THEOREM *Let $x_0 \in A \subset X$ and assume that X is path connected and (X,A) is n-connected for some $n \geq 0$. Then the inclusion map $\Delta(X,A,x_0)^n \subset \Delta(X)$ is a chain equivalence.*

PROOF For σ: $\Delta^q \to X$ we define $P(\sigma)$ by induction on q to satisfy the properties of lemma 7, and to have the additional property that if σ is in $\Delta(X,A,x_0)^n$, then $P(\sigma)$ is the composite

$$\Delta^q \times I \xrightarrow{p} \Delta^q \xrightarrow{\sigma} X$$

where p is projection to the first factor.

If $q = 0$, then σ: $\Delta^0 \to X$ is a point of X, and because X is path connected, there is a map $P(\sigma)$: $\Delta^0 \times I \to X$ such that $P(\sigma)(\Delta^0 \times 0) = \sigma(\Delta^0)$ and $P(\sigma)(\Delta^0 \times 1) = x_0$ [and if $\sigma(\Delta^0) = x_0$, we take $P(\sigma)$ to be the constant map to x_0]. This defines $P(\sigma)$ for all σ of degree 0 to have the desired properties.

Assume $0 < q \leq n$ and that $P(\sigma)$ has been defined for all σ of degree $< q$ to have the properties stated above. Given a singular simplex σ: $\Delta^q \to X$, if σ is in $\Delta(X,A,x_0)^n$, define $P(\sigma) = \sigma \circ p$. If σ is not in $\Delta(X,A,x_0)^n$, (*a*) and (*c*) of lemma 7 define $P(\sigma)$ on $\Delta^q \times 0 \cup \dot\Delta^q \times I$, and we let f: $\Delta^q \times 0 \cup \dot\Delta^q \times I \to X$ be this map. There is a homeomorphism h: $E^q \times I \approx \Delta^q \times I$ such that

$$h(E^q \times 0) = \Delta^q \times 0 \cup \dot\Delta^q \times I, \quad h(S^{q-1} \times 0) = \dot\Delta^q \times 1$$

and

$$h(S^{q-1} \times I \cup E^q \times 1) = \Delta^q \times 1$$

Let $f'\colon (E^q,S^{q-1}) \to (X,A)$ be defined by $f'(z) = f(h(z,0))$. Because $q \le n$ and (X,A) is n-connected, there is a homotopy $H\colon (E^q,S^{q-1}) \times I \to (X,A)$ from f' to some map of E^q into A (in fact, by the definition of n-connectedness, there is even such a homotopy relative to S^{q-1}). Then the composite

$$\Delta^q \times I \xrightarrow{h^{-1}} E^q \times I \xrightarrow{H} X$$

can be taken as $P(\sigma)$.

In this way $P(\sigma)$ is defined for all degrees $q \le n$. Note that a singular simplex of degree $> n$ is in $\Delta(X,A,x_0)^n$ if and only if every proper face is in $\Delta(X,A,x_0)^n$. Therefore, if $P(\sigma)$ has been defined for all degrees $< q$, where $q > n$, and if $\sigma\colon \Delta^q \to X$, then we define $P(\sigma) = \sigma \circ p$ if σ is in $\Delta(X,A,x_0)^n$ and to be any map $\Delta^q \times I \to X$ satisfying (*a*) and (*c*) of lemma 7 (such maps exist by the homotopy extension property). Then $P(\sigma)$ will necessarily satisfy (*b*) of lemma 7, and we have shown that $P(\sigma)$ can be defined for all σ to satisfy lemma 7. ∎

For $n \ge 0$ we define

$$H_q^{(n)}(X,A,x_0) = H_q(\Delta(X,A,x_0)^n,\ \Delta(X,A,x_0)^n \cap \Delta(A))$$

There are canonical homomorphisms

$$\cdots \to H_q^{(n)}(X,A,x_0) \to H_q^{(n-1)}(X,A,x_0) \to \cdots \to H_q^{(0)}(X,A,x_0) \to H_q(X,A)$$

9 COROLLARY *Assume that A is path connected and for some $n \ge 0$, (X,A) is n-connected. Then the canonical map is an isomorphism for all q*

$$H_q^{(n)}(X,A,x_0) \approx H_q(X,A)$$

PROOF For any $n \ge 0$, $\Delta(X,A,x_0)^n \cap \Delta(A)$ is generated by the set of singular simplexes of A all of whose vertices are at x_0. This is independent of n, and because A is path connected, $(A,\{x_0\})$ is 0-connected, and it follows from theorem 8 that the inclusion map $\Delta(X,A,x_0)^n \cap \Delta(A) \subset \Delta(A)$ is a chain equivalence for all $n \ge 0$.

Since (X,A) is n-connected, where $n \ge 0$, and A is path connected, X is also path connected, and by theorem 8, the inclusion map $\Delta(X,A,x_0)^n \subset \Delta(X)$ is a chain equivalence. The result follows from these facts, using exactness and the five lemma. ∎

5 THE HUREWICZ ISOMORPHISM THEOREM

The main result of this section asserts that if X and A are path connected and for some $n \ge 1$, (X,A) is n-connected, then the Hurewicz homomorphism φ induces an isomorphism φ' of $\pi'_{n+1}(X,A,x_0)$ with $H_{n+1}(X,A)$. This result is equivalent to a homotopy addition theorem which asserts that the sum of the $(n+1)$-dimensional faces of an $(n+2)$-simplex is the homotopy boundary of the identity map of the simplex. We prove both these theorems simultaneously by induction on n.

In the proof we shall make essential use of the complexes $\Delta(X,A,x_0)^n$ and of corollary 7.4.9. Let α: $(\Delta^n,\dot\Delta^n,(\Delta^n)^0) \to (X,A,x_0)$ represent an element of $\pi_n(X,A,x_0)$. Then α is a singular simplex in $\Delta(X,A,x_0)^{n-1}$ and represents a homology class $\{\alpha\} \in H_n{}^{(n-1)}(X,A,x_0)$. Since any element of $\pi_n(X,A,x_0)$ can be represented by such a map α, the Hurewicz homomorphism φ': $\pi'_n(X,A,x_0) \to H_n(X,A)$ factors into the composite

$$\pi'_n(X,A,x_0) \xrightarrow{\varphi''} H_n{}^{(n-1)}(X,A,x_0) \to H_n(X,A)$$

and there is a commutative diagram

$$\begin{tikzcd} \pi_n(X,A,x_0) \arrow[r, "\eta"] \arrow[d, "\varphi"] \arrow[dr, "\varphi'"] & \pi'_n(X,A,x_0) \arrow[d, "\varphi''"] \\ H_n(X,A) & H_n{}^{(n-1)}(X,A,x_0) \arrow[l] \end{tikzcd}$$

We now formulate the propositions corresponding to the relative and absolute Hurewicz isomorphism theorems.

1 PROPOSITION Φ_n $(n \geq 2)$. *Let A be path connected and let (X,A) be $(n-1)$-connected. Then φ' is an isomorphism*

$$\varphi'\colon \pi'_n(X,A,x_0) \approx H_n(X,A)$$

2 PROPOSITION $\bar\Phi_n$ $(n \geq 1)$. *Let X be $(n-1)$-connected. Then φ' is an isomorphism*

$$\varphi'\colon \pi'_n(X,x_0) \approx H_n(X,x_0)$$

We shall prove both these propositions simultaneously by induction on n, together with a third proposition, which we now formulate. For $n \geq 2$, each face map e^i_{n+1} is a map of triples

$$e^0_{n+1}\colon (\Delta^n,\dot\Delta^n,v_0) \to (\dot\Delta^{n+1},(\Delta^{n+1})^{n-1},v_1)$$
$$e^i_{n+1}\colon (\Delta^n,\dot\Delta^n,v_0) \to (\dot\Delta^{n+1},(\Delta^{n+1})^{n-1},v_0) \qquad 0 < i \leq n+1$$

For vertices v and v' of Δ^{n+1} we use $[vv']$ to denote the path class of the linear path in Δ^{n+1} from v to v'. We define an element $b_1 \in \pi_1(\dot\Delta^2,v_0)$ and, for $n \geq 2$, an element $b_n \in \pi_n(\dot\Delta^{n+1},(\Delta^{n+1})^{n-1},v_0)$ by

$$b_1 = [v_0v_1] * [v_1v_2] * [v_2v_0]$$
$$b_2 = (h_{[v_0v_1]}[e_3{}^0])[e_3{}^2][e_3{}^1]^{-1}[e_3{}^3]^{-1}$$
$$b_n = h_{[v_0v_1]}[e^0_{n+1}] + \sum_{0<i\leq n+1} (-1)^i[e^i_{n+1}] \qquad n \geq 3$$

For $n = 1$ let j: $(\dot\Delta^2,v_0) \subset (\Delta^2,v_0)$ and for $n \geq 2$ let j: $(\dot\Delta^{n+1},(\Delta^{n+1})^{n-1},v_0) \subset (\Delta^{n+1},(\Delta^{n+1})^{n-1},v_0)$. The following proposition corresponds to the homotopy addition theorem.

3 PROPOSITION B_n $(n \geq 1)$. $j_\# b_n = 0$.

The simultaneous proof of propositions 1, 2, and 3 will consist of the following five parts:

(*a*) Proof of B_1
(*b*) Proof that $B_1 \Rightarrow \bar{\Phi}_1$
(*c*) Proof that $\bar{\Phi}_1, \bar{\Phi}_2, \ldots, \bar{\Phi}_{n-1} \Rightarrow B_n$ for $n \geq 2$
(*d*) Proof that $B_n \Rightarrow \Phi_n$ for $n \geq 2$
(*e*) Proof that $\Phi_n \Rightarrow \bar{\Phi}_n$ for $n \geq 2$

(*a*) **PROOF OF** B_1 We must prove that $j_\# b_1 = 0$. But $j_\# b_1 \in \pi_1(\Delta^2,v_0)$, and $\pi_1(\Delta^2,v_0) = 0$ because Δ^2 is contractible. ∎

(*b*) **PROOF THAT** $B_1 \Rightarrow \bar{\Phi}_1$ Let X be path connected. We must prove that $\varphi'\colon \pi_1'(X,x_0) \approx H_1(X,x_0)$. Because X is path connected, the inclusion map $\Delta(X,\{x_0\},x_0)^0 \subset \Delta(X)$ is a chain equivalence, and we need only show that

$$\varphi''\colon \pi_1'(X,x_0) \approx H_1{}^{(0)}(X,\{x_0\},x_0)$$

If $\alpha\colon (\Delta^1,\dot{\Delta}^1) \to (X,x_0)$ represents an element $[\alpha]' \in \pi_1'(X,x_0)$, then $\varphi''[\alpha]' = \{\alpha\}$, where $\{\alpha\}$ is the homology class in $H_1{}^{(0)}(X,\{x_0\},x_0)$ of the singular cycle α. Given a singular 1-simplex $\sigma\colon (\Delta^1,\dot{\Delta}^1) \to (X,x_0)$ in $\Delta(X,\{x_0\},x_0)^0$, it determines an element $[\sigma] \in \pi_1(X,x_0)$, and therefore an element $[\sigma]' \in \pi_1'(X,x_0)$. If σ is the constant singular 1-simplex at x_0, then clearly, $[\sigma]' = 0$. Because $\pi_1'(X,x_0)$ is abelian and $\Delta_1(X,\{x_0\},x_0)^0$ is the free abelian group generated by the singular simplexes in it, there is a homomorphism

$$\psi\colon \Delta_1(X,\{x_0\},x_0)^0/\Delta_1(x_0) \to \pi_1'(X,x_0)$$

such that $\psi(\sigma) = [\sigma]'$. We shall show, by using B_1, that the composite

$$\Delta_2(X,\{x_0\},x_0)^0/\Delta_2(x_0) \xrightarrow{\partial} \Delta_1(X,\{x_0\},x_0)^0/\Delta_1(x_0) \xrightarrow{\psi} \pi_1'(X,x_0)$$

is trivial. Given $\sigma\colon (\Delta^2,(\Delta^2)^0) \to (X,x_0)$, let $\sigma^{(0)}$, $\sigma^{(1)}$, and $\sigma^{(2)}$ be the faces of σ, as usual. Then

$$\begin{aligned}\psi\partial[\sigma] &= [\sigma^{(2)}]' + [\sigma^{(0)}]' - [\sigma^{(1)}]' = [(\sigma^{(2)} * \sigma^{(0)}) * (\sigma^{(1)})^{-1}]' \\ &= \eta(\sigma \mid \dot{\Delta}^2)_\#([v_0v_1] * [v_1v_2] * [v_2v_0]) = \eta\sigma_\# j_\# b_1 = 0\end{aligned}$$

Therefore ψ defines a homomorphism

$$\psi'\colon H_1{}^{(0)}(X,\{x_0\},x_0) \to \pi_1'(X,x_0)$$

and this is easily seen to be an inverse of φ''. ∎

(*c*) **PROOF THAT** $\bar{\Phi}_1, \ldots, \bar{\Phi}_{n-1} \Rightarrow B_n$ **FOR** $n \geq 2$ Consider the commutative diagram

$$\begin{tikzcd}
\pi_{n+1}(\Delta^{n+1},\dot{\Delta}^{n+1},v_0) \arrow[dr, "\partial'"] \arrow[dd, "\partial"'] & & \pi_n(\Delta^{n+1},(\Delta^{n+1})^{n-1},v_0) \\
& \pi_n(\dot{\Delta}^{n+1},(\Delta^{n+1})^{n-1},v_0) \arrow[ur, "j_\#"] \arrow[dr, "\partial''"] & \\
\pi_n(\dot{\Delta}^{n+1},v_0) \arrow[ur, "i_\#"'] & & \pi_{n-1}((\Delta^{n+1})^{n-1},v_0)
\end{tikzcd}$$

The top row, being part of the homotopy sequence of the triple $(\Delta^{n+1},\dot{\Delta}^{n+1},(\Delta^{n+1})^{n-1})$, is exact. The bottom row, being part of the homotopy sequence of the pair $(\Delta^{n+1},(\dot\Delta^{n+1})^{n-1})$, is also exact. From the exactness of the homotopy sequence of the pair $(\Delta^{n+1},\dot\Delta^{n+1})$ and the fact that Δ^{n+1} is contractible, it follows that ∂ is an isomorphism. Therefore

$$\ker j_{\#} = \operatorname{im} \partial' = \operatorname{im} (i_{\#} \circ \partial) = \operatorname{im} i_{\#} = \ker \partial''$$

Thus B_n is equivalent to the equation $\partial''(b_n) = 0$. We prove the latter, giving one proof for $n = 2$ and another for $n > 2$.

If $n = 2$, we have

$$\partial''(b_2) = (h_{[v_0v_1]}\partial''[e_3{}^0])\partial''[e_3{}^2]\partial''[e_3{}^1]^{-1}\partial''[e_3{}^3]^{-1}$$

To calculate $\partial''[e_3{}^i]$, let ξ: $(\Delta^2,\dot\Delta^2,v_0) \subset (\Delta^2,\dot\Delta^2,v_0)$ be the identity map. Then $[\xi] \in \pi_2(\Delta^2,\dot\Delta^2,v_0)$, and because $\pi_1(\dot\Delta^2,v_0)$ is infinite cyclic (since $\dot\Delta^2$ is homeomorphic to S^1), it follows from $\bar\Phi_1$ that φ: $\pi_1(\dot\Delta^2,v_0) \approx H_1(\dot\Delta^2,v_0)$. There is a commutative square

$$\begin{array}{ccc} \pi_2(\Delta^2,\dot\Delta^2,v_0) & \xrightarrow{\partial} & \pi_1(\dot\Delta^2,v_0) \\ \varphi\downarrow & & \approx\downarrow\varphi \\ H_2(\Delta^2,\dot\Delta^2) & \xrightarrow{\partial} & H_1(\dot\Delta^2,v_0) \end{array}$$

and $$\partial\varphi[\xi] = \partial\{\xi\} = \{\xi^{(2)} + \xi^{(0)} - \xi^{(1)}\} = \{\omega\} = \varphi[\omega]$$

where ω: $(\Delta^1,\dot\Delta^1) \to (\dot\Delta^2,v_0)$ is the path $\omega = (\xi^{(2)} * \xi^{(0)}) * (\xi^{(1)})^{-1}$. (The 2-chain $\xi^{(2)} + \xi^{(0)} - \xi^{(1)}$ is homologous to ω because it is easy to find singular 2-simplexes σ_1 and σ_2 in $\dot\Delta^2$ such that

$$\begin{array}{lll} \sigma_1{}^{(0)} = \xi^{(0)} & \sigma_1{}^{(1)} = \xi^{(2)} * \xi^{(0)} & \sigma_1{}^{(2)} = \xi^{(2)} \\ \sigma_2{}^{(0)} = \xi^{(1)} & \sigma_2{}^{(1)} = \xi^{(2)} * \xi^{(0)} & \sigma_2{}^{(2)} = (\xi^{(2)} * \xi^{(0)}) * (\xi^{(1)})^{-1} \end{array}$$

Then $\partial(\sigma_1 - \sigma_2) = \xi^{(2)} + \xi^{(0)} - \xi^{(1)} - \omega$.) Because φ is an isomorphism, it follows that

$$\partial[\xi] = [\omega] = [v_0v_1] * [v_1v_2] * [v_2v_0]$$

To return to the calculation of $\partial''[e_3{}^i]$, we have

$$\begin{aligned} \partial''[e_3{}^i] &= \partial''(e_3{}^i)_{\#}[\xi] = (e_3{}^i \mid \dot\Delta^2)_{\#}\partial[\xi] \\ &= [e_3{}^i(v_0)e_3{}^i(v_1)] * [e_3{}^i(v_1)\, e_3{}^i(v_2)] * [e_3{}^i(v_2)\, e_3{}^i(v_0)] \end{aligned}$$

Using this, direct substitution into the right-hand side of the equation for $\partial''(b_2)$ shows that $\partial''(b_2) = 0$.

For $n > 2$ note that $(\dot\Delta^{n+1})^{n-1}$ contains the two-dimensional skeleton of Δ^{n+1}. Therefore $(\dot\Delta^{n+1})^{n-1}$ is simply connected (because Δ^{n+1} is simply connected). Similarly, for $q \le n - 2$, $H_q((\dot\Delta^{n+1})^{n-1},v_0) \approx H_q(\Delta^{n+1},v_0) = 0$. By $\bar\Phi_1, \ldots, \bar\Phi_{n-2}$, it follows that $(\dot\Delta^{n+1})^{n-1}$ is $(n - 2)$-connected, and by $\bar\Phi_{n-1}$, there is an isomorphism

$$\varphi\colon \pi_{n-1}((\dot\Delta^{n+1})^{n-1},v_0) \approx H_{n-1}((\dot\Delta^{n+1})^{n-1},v_0)$$

Hence, to complete the proof it suffices to show that $\varphi\partial''(b_n) = 0$. This follows from the equalities

$$\varphi\partial''(b_n) = \partial''\varphi(b_n) = \partial''\{\Sigma\,(-1)^i e^i_{n+1}\} = \partial''\partial'\{\xi_{n+1}\} = \partial'' i_* \partial\{\xi_{n+1}\} = 0 \quad \blacksquare$$

(d) **PROOF THAT** $B_n \Rightarrow \Phi_n$ **FOR** $n \geq 2$ The argument is similar to the proof of part (b) above. The map φ' factors into the composite

$$\pi'_n(X,A,x_0) \xrightarrow{\varphi''} H_n{}^{(n-1)}(X,A,x_0) \underset{\approx}{\rightarrow} H_n(X,A)$$

If α: $(\Delta^n,\dot\Delta^n,v_0) \to (X,A,x_0)$ is a map such that α maps all the vertices to x_0, then $\varphi''[\alpha]' = \{\alpha\} \in H_n{}^{(n-1)}(X,A,x_0)$. To define an inverse of φ'', if σ: $(\Delta^n,\dot\Delta^n,(\Delta^n)^0) \to (X,A,x_0)$ is a singular simplex in $\Delta_n(X,A,x_0)^{n-1}$, then $[\sigma] \in \pi_n(X,A,x_0)$ and $\eta[\sigma] = [\sigma]' \in \pi'_n(X,A,x_0)$. If $\sigma(\Delta^n) \subset A$, then $[\sigma]' = 0$, and because $\pi'_n(X,A,x_0)$ is abelian, there is a homomorphism

$$\psi\colon \Delta_n(X,A,x_0)^{n-1}/(\Delta_n(X,A,x_0)^{n-1} \cap \Delta_n(A)) \to \pi'_n(X,A,x_0)$$

such that $\psi(\sigma) = [\sigma]'$.

We show that the composite

$$\psi \circ \partial\colon \Delta_{n+1}(X,A,x_0)^{n-1}/(\Delta_{n+1}(X,A,x_0)^{n-1} \cap \Delta_{n+1}(A)) \to \pi'_n(X,A,x_0)$$

is trivial. This follows from B_n, because if

$$\sigma\colon (\Delta^{n+1},(\Delta^{n+1})^{n-1},(\Delta^{n+1})^0) \to (X,A,x_0)$$

then

$$\begin{aligned}\psi\partial(\sigma) &= \Sigma\,(-1)^i[\sigma^{(i)}]' = \eta(\sigma \mid (\dot\Delta^{n+1},(\Delta^{n+1})^{n-1})_\#(b_n))\\ &= \eta\sigma_\# j_\#(b_n) = 0\end{aligned}$$

Therefore ψ defines a homomorphism

$$\psi'\colon H_n{}^{(n-1)}(X,A,x_0) \to \pi'_n(X,A,x_0)$$

such that $\psi'\{\sigma\} = [\sigma]'$, and ψ' is easily seen to be an inverse of φ''. $\blacksquare$

(e) **PROOF THAT** $\Phi_n \Rightarrow \bar\Phi_n$ **FOR** $n \geq 2$ For $n \geq 2$, if X is $(n-1)$-connected, then the pair $(X,\{x_0\})$ is $(n-1)$-connected and $\pi'_n(X,\{x_0\},x_0)$ is canonically isomorphic to $\pi'_n(X,x_0) = \pi_n(X,x_0)$. Then $\bar\Phi_n$ results from Φ_n applied to the pair $(X,\{x_0\})$. $\blacksquare$

This completes the proof of propositions 1, 2, and 3. From proposition 1 we obtain the following *relative Hurewicz isomorphism theorem.*

4 THEOREM *Let $x_0 \in A \subset X$ and assume that A and X are path connected. If there is an $n \geq 2$ such that $\pi_q(X,A,x_0) = 0$ for $q < n$, then $H_q(X,A) = 0$ for $q < n$ and φ' is an isomorphism*

$$\varphi'\colon \pi'_n(X,A,x_0) \approx H_n(X,A)$$

Conversely, if A and X are simply connected and there is an $n \geq 2$ such that $H_q(X,A) = 0$ for $q < n$, then $\pi_q(X,A,x_0) = 0$ for $q < n$ and φ is an isomorphism

$$\varphi\colon \pi_n(X,A,x_0) \approx H_n(X,A) \quad \blacksquare$$

Similarly, from proposition 2 we obtain the following *absolute Hurewicz isomorphism theorem.*

5 THEOREM *Let $x_0 \in X$ and assume that there is $n \geq 1$ such that $\pi_q(X,x_0) = 0$ for $q < n$. Then $H_q(X,x_0) = 0$ for $q < n$ and φ' is an isomorphism*

$$\varphi'\colon \pi_n'(X,x_0) \approx H_n(X,x_0)$$

Conversely, if X is simply connected and there is $n \geq 2$ such that $H_q(X,x_0) = 0$ for $q < n$, then $\pi_q(X,x_0) = 0$ for $q < n$ and φ is an isomorphism

$$\varphi\colon \pi_n(X,x_0) \approx H_n(X,x_0) \quad \blacksquare$$

In the absolute case when X is simply connected and in the relative case when X and A are simply connected, each of these theorems asserts that the first nonvanishing homotopy group is isomorphic to the first nonvanishing homology group.

6 COROLLARY *For $n \geq 1$ there is a commutative diagram of isomorphisms*

$$\begin{array}{ccc}
\pi_{n+1}(E^{n+1},S^n,p_0) & \xrightarrow{\partial} & \pi_n(S^n,p_0) \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\
H_{n+1}(E^{n+1},S^n) & \xrightarrow{\partial} & H_n(S^n,p_0)
\end{array}$$

PROOF The diagram is commutative, by theorem 7.4.3*a*, and both horizontal maps are isomorphisms because E^{n+1} is contractible [and because the homotopy and homology sequences of (E^{n+1},S^n,p_0) are exact]. The right-hand vertical map is an isomorphism, by proposition 2 and the fact that (in the case $n = 1$) $\pi_1(S^1,p_0)$ is abelian. $\blacksquare$

The following useful consequence of corollary 6 is called the *Brouwer degree theorem.*

7 COROLLARY *For $n \geq 1$ two maps f, $g\colon S^n \to S^n$ are homotopic if and only if $f_* = g_*\colon H_n(S^n) \to H_n(S^n)$. Similarly, two maps f, $g\colon (E^{n+1},S^n) \to (E^{n+1},S^n)$ are homotopic if and only if $f_* = g_*\colon H_{n+1}(E^{n+1},S^n) \to H_{n+1}(E^{n+1},S^n)$.*

PROOF We consider the absolute case first. Given maps f, $g\colon S^n \to S^n$, there exist homotopic maps f' and g', respectively, such that $f'(p_0) = g'(p_0) = p_0$ (because S^n is path connected). Because S^n is n-simple, f' and g' are freely homotopic if and only if they are homotopic as maps from (S^n,p_0) to (S^n,p_0). Therefore $f \simeq g$ if and only if $[f'] = [g']$ in $\pi_n(S^n,p_0)$. By corollary 6, $[f'] = [g']$ if and only if $\varphi[f'] = \varphi[g']$, and from the definition of φ, $\varphi[f'] = \varphi[g']$ if and only if

$$f_*' = g_*'\colon H_n(S^n,p_0) \to H_n(S^n,p_0)$$

Since there are commutative squares

$$\begin{array}{ccc}
H_n(S^n) & \xrightarrow{\approx} & H_n(S^n,p_0) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*'} \\
H_n(S^n) & \xrightarrow{\approx} & H_n(S^n,p_0)
\end{array}
\qquad\qquad
\begin{array}{ccc}
H_n(S^n) & \xrightarrow{\approx} & H_n(S^n,p_0) \\
{\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*'} \\
H_n(S^n) & \xrightarrow{\approx} & H_n(S^n,p_0)
\end{array}$$

the result follows.

For the relative case note that because E^{n+1} is contractible, it follows from the homotopy extension property of (E^{n+1},S^n) that two maps $f, g\colon (E^{n+1},S^n) \to (E^{n+1},S^n)$ are homotopic if and only if $f \mid S^n$, $g \mid S^n\colon S^n \to S^n$ are homotopic. Since there are commutative squares

$$\begin{array}{ccc} H_{n+1}(E^{n+1},S^n) & \xrightarrow[\approx]{\partial} & H_n(S^n) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f\mid S^n)_*} \\ H_{n+1}(E^{n+1},S^n) & \xrightarrow[\approx]{\partial} & H_n(S^n) \end{array} \qquad \begin{array}{ccc} H_{n+1}(E^{n+1},S^n) & \xrightarrow[\approx]{\partial} & H_n(S^n) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle (g\mid S^n)_*} \\ H_{n+1}(E^{n+1},S^n) & \xrightarrow[\approx]{\partial} & H_n(S^n) \end{array}$$

the relative case follows from the absolute case. ■

8 COROLLARY *For $x_0 \in X$ the map*

$$\psi\colon [S^n,p_0;\, X,x_0] \to \operatorname{Hom}\,(\pi_n(S^n,p_0),\, \pi_n(X,x_0))$$

sending $[\alpha]$ to $\alpha_\#$ is an isomorphism.

PROOF This follows from corollary 6, because the fact that $\pi_n(S^n,p_0)$ is infinite cyclic implies that there is an isomorphism

$$\beta\colon \operatorname{Hom}\,(\pi_n(S^n,p_0),\, \pi_n(X,x_0)) \approx \pi_n(X,x_0)$$

sending a homomorphism λ to $\lambda(a)$, where $a \in \pi_n(S^n,p_0)$ is the homotopy class of the identity map. Then, $(\beta \circ \psi)[\alpha] = \alpha_\#(a) = [\alpha]$, and so ψ is an isomorphism. ■

The following useful consequence of the relative Hurewicz isomorphism theorem is known as the *Whitehead theorem.*

9 THEOREM *Let X and Y be path-connected pointed spaces and let $f\colon (X,x_0) \to (Y,y_0)$ be a map. If there is $n \geq 1$ such that*

$$f_\#\colon \pi_q(X,x_0) \to \pi_q(Y,y_0)$$

is an isomorphism for $q < n$ and an epimorphism for $q = n$, then

$$f_*\colon H_q(X,x_0) \to H_q(Y,y_0)$$

is an isomorphism for $q < n$ and an epimorphism for $q = n$. Conversely, if X and Y are simply connected and f_* is an isomorphism for $q < n$ and an epimorphism for $q = n$, then $f_\#$ is an isomorphism for $q < n$ and an epimorphism for $q = n$.

PROOF Let Z be the mapping cylinder of f. There are inclusion maps $i\colon X \subset Z$ and $j\colon Y \subset Z$ and a deformation retraction $r\colon Z \to Y$ such that $f = r \circ i$. Then $r\colon (Z,y_0) \to (Y,y_0)$ induces isomorphisms $r_\#\colon \pi_q(Z,y_0) \approx \pi_q(Y,y_0)$ and $r_*\colon H_q(Z,y_0) \approx H_q(Y,y_0)$ for all q. Because X and Y are path connected, so is Z, and $\pi_q(Z,x_0) \approx \pi_q(Z,y_0)$. Therefore $r\colon (Z,x_0) \to (Y,y_0)$ also induces isomorphisms $r_\#\colon \pi_q(Z,x_0) \approx \pi_q(Y,y_0)$ and $r_*\colon H_q(Z,x_0) \approx H_q(Y,y_0)$ for all q. It follows that we can replace (Y,y_0) in the theorem by (Z,x_0) and the conditions on $f_\#$ and f_* by the corresponding conditions on $i_\#$ and i_*. From the exactness of the homotopy sequence of (Z,X,x_0), it follows that $i_\#$ is an isomorphism for $q < n$ and an epimorphism for $q = n$ if and only if $\pi_q(Z,X,x_0) = 0$ for $q \leq n$. Similarly, from the exactness of the homology sequence of the triple (Z,X,x_0), it follows that i_* is an isomorphism for $q < n$ and an epimorphism for $q = n$ if and only if $H_q(Z,X) = 0$ for $q \leq n$. The result now follows from the relative Hurewicz isomorphism theorem 4. ∎

6 CW COMPLEXES

For homotopy theory the most tractable family of topological spaces seems to be the family of *CW* complexes (or the family of spaces each having the same homotopy type as a *CW* complex). *CW* complexes are built in stages, each stage being obtained from the preceding by adjoining cells of a given dimension. The cellular structure of such a complex bears a direct connection with its homotopy properties. Even for such nice spaces as polyhedra it is useful to consider representations of them as *CW* complexes, because such complexes will frequently require fewer cells than a simplicial triangulation.

In this section we shall investigate *CW* complexes and related concepts. In Sec. 7.8 we shall show that any topological space can be approximated by a *CW* complex which is unique up to homotopy. We begin with some results about a space X obtained from a subspace A by adjoining n-cells (defined in Sec. 3.8).

1 LEMMA *If X is obtained from A by adjoining n-cells, then $X \times 0 \cup A \times I$ is a strong deformation retract of $X \times I$.*

PROOF For each n-cell e_j^n of $X - A$ let

$$f_j\colon (E^n,S^{n-1}) \to (e_j^n,\dot{e}_j^n)$$

be a characteristic map. Let $D\colon (E^n \times I) \times I \to E^n \times I$ be a strong deformation retraction of $E^n \times I$ to $E^n \times 0 \cup S^{n-1} \times I$ (which exists, by corollary 3.2.4). There is a well-defined map $D_j\colon (e_j^n \times I) \times I \to e_j^n \times I$ characterized by the equation

$$D_j((f_j(z),t),t') = (f_j \times 1_I)(D(z,t,t')) \qquad z \in E^n;\ t,\ t' \in I$$

Then there is a map $D'\colon (X \times I) \times I \to X \times I$ such that $D' \mid (e_j \times I) \times I = D_j$ and $D'(a,t,t') = (a,t)$ for $a \in A$, and $t,\ t' \in I$, and D' is a strong deformation retraction of $X \times I$ to $X \times 0 \cup A \times I$. ∎

2 COROLLARY *If X is obtained from A by adjoining n-cells, then the inclusion map $A \subset X$ is a cofibration.* ∎

3 LEMMA *Let X be obtained from A by adjoining n-cells and let (Y,B) be a pair such that $\pi_n(Y,B,b) = 0$ for all $b \in B$ if $n \geq 1$ and such that every point of Y can be joined to B by a path if $n = 0$. Then any map from (X,A) to (Y,B) is homotopic relative to A to a map from X to B.*

PROOF This follows from theorem 7.2.1 by a technique similar to that in lemma 1 above. ■

A *relative CW complex* (X,A) consists of a topological space X, a closed subspace A, and a sequence of closed subspaces $(X,A)^k$ for $k \geq 0$ such that

(*a*) $(X,A)^0$ is obtained from A by adjoining 0-cells.
(*b*) For $k \geq 1$, $(X,A)^k$ is obtained from $(X,A)^{k-1}$ by adjoining k-cells.
(*c*) $X = \cup (X,A)^k$.
(*d*) X has a topology coherent with $\{(X,A)^k\}_k$.

In this case $(X,A)^k$ is called the *k-skeleton of X relative to A*. If $X = (X,A)^n$ for some n, then we say *dimension* $(X - A) \leq n$. An *absolute CW complex* X is a relative CW complex $(X, \varnothing)$, and its k-skeleton is denoted by X^k.

Following are a number of examples.

4 If (K,L) is a simplicial pair, there is a relative CW complex $(|K|,|L|)$, with $(|K|,|L|)^k = |K^k \cup L|$.

5 If (X,A) is a relative CW complex, for any k the pair $(X, (X,A)^k)$ is a relative CW complex, with

$$(X, (X,A)^k)^q = \begin{cases} (X,A)^k & q \leq k \\ (X,A)^q & q > k \end{cases}$$

Similarly, the pair $((X,A)^k, A)$ is a relative CW complex, with

$$((X,A)^k, A)^q = \begin{cases} (X,A)^q & q \leq k \\ (X,A)^k & q > k \end{cases}$$

6 As in example 3.8.7, for $i = 1, 2$, or 4 let F_i be $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{Q}$, respectively, and for $q \geq 0$ let $P_q(F_i)$ be the corresponding projective space of dimension q over F_i. Then $P_q(F_i)$ is a CW complex, with

$$(P_q(F_i))^k = \begin{cases} P_{[k/i]}(F_i) & k \leq iq \\ P_q(F_i) & k > iq \end{cases}$$

7 E^n is a CW complex, with $(E^n)^k = p_0$ for $k < n - 1$, $(E^n)^{n-1} = S^{n-1}$, and $(E^n)^k = E^n$ for $k \geq n$.

8 I is a CW complex, with $(I)^0 = \dot{I}$ and $(I)^k = I$ for $k \geq 1$.

9 If (X,A) and (Y,B) are relative CW complexes and either X or Y is locally compact, then $(X,A) \times (Y,B)$ is also a CW complex,[1] with

$$((X,A) \times (Y,B))^k = \cup_{i+j=k} (X,A)^i \times (Y,B)^j \cup X \times B \cup A \times Y$$

10 If (X,A) is a relative CW complex, so is $(X,A) \times I$, with

$$(X \times I, A \times I)^k = (X,A)^k \times \dot{I} \cup (X,A)^{k-1} \times I \cup A \times I$$

[1] It is not true that the product of two CW complexes is always a CW complex. For a counterexample, see C. H. Dowker, Topology of metric complexes, *American Journal of Mathematics*, vol. 74, pp. 555–577, 1952.

11 If (X,A) is a relative CW complex, then X/A is a CW complex, with $(X/A)^k = (X,A)^k/A$.

A *subcomplex* (Y,B) of a relative CW complex (X,A) is a relative CW complex such that Y is a closed subset of X and $(Y,B)^k = Y \cap (X,A)^k$ for all k. If (Y,B) is a subcomplex of (X,A), then $(X, A \cup Y)$ is a relative CW complex, with $(X, A \cup Y)^k = (X,A)^k \cup Y$ for all k. In particular, if X is a CW complex and A is a subcomplex of X, then (X,A) is a relative CW complex. A CW *pair* (X,A) consists of a CW complex X and subcomplex A (hence a CW pair is a relative CW complex).

The definition of relative CW complex suggests its inductive construction. We start with a space A, attach 0-cells to A to obtain a space A_0, attach 1-cells to A_0 to obtain A_1, and continue in this way to define A_k for all $k \geq 0$. Letting X be the space obtained by topologizing $\cup A_k$ with the topology coherent with $\{A_k\}_{k\geq 0}$, then (X,A) is a relative CW complex, with $(X,A)^k = A_k$.

12 THEOREM *If (X,A) is a relative CW complex, then the inclusion map $A \subset X$ is a cofibration.*

PROOF This follows from corollary 2, using induction and the fact that $X \times I$ has the topology coherent with $\{(X,A)^k \times I\}_k$. ∎

13 THEOREM *Let (X,A) be a relative CW complex, with dimension $(X - A) \leq n$, and let (Y,B) be n-connected. Then any map from (X,A) to (Y,B) is homotopic relative to A to a map from X to B.*

PROOF This follows, using induction, from corollary 7.2.2, lemma 3, and theorem 12. ∎

14 COROLLARY *Let (X,A) be a relative CW complex and let (Y,B) be n-connected for all n. Then any map from (X,A) to (Y,B) is homotopic relative to A to a map from X to B.*

PROOF Let $f\colon (X,A) \to (Y,B)$ be a map. It follows from theorems 12 and 13 that there is a sequence of homotopies

$$H_k\colon (X,A) \times I \to (Y,B) \qquad k \geq 0$$

constructed by induction on k such that

(*a*) $H_0(x,0) = f(x)$ for $x \in X$.
(*b*) $H_k(x,1) = H_{k+1}(x,0)$ for $x \in X$.
(*c*) H_k is a homotopy relative to $(X,A)^{k-1}$.
(*d*) $H_k((X,A)^k \times 1) \subset B$.

Then a homotopy $H\colon (X,A) \times I \to (Y,B)$ with the required properties is defined by

$$H(x,t) = H_{k-1}\left(x, \frac{t - (1 - 1/k)}{(1/k) - 1/(k+1)}\right) \qquad 1 - \frac{1}{k} \leq t \leq 1 - \frac{1}{k+1}$$

$$H(x,1) = H_k(x,1) \qquad x \in (X,A)^k \quad ∎$$

15 LEMMA *If X is obtained from A by adjoining n-cells, then for $n \geq 1$, (X,A) is $(n - 1)$-connected.*

PROOF For $k \leq n - 1$ let $f\colon (E^k,S^{k-1}) \to (X,A)$ be a map. Because $f(E^k)$ is compact, there exist a finite number, say, $e_1, \ldots, e_m$, of n-cells of $X - A$ such that $f(E^k) \subset e_1 \cup \cdots \cup e_m \cup A$. For $1 \leq i \leq m$ let x_i be a point of $e_i - \dot{e}_i$. Each of the sets $Y = A \cup (e_1 - x_1) \cup \cdots \cup (e_m - x_m)$ and $e_i - \dot{e}_i$ for $1 \leq i \leq m$ intersects $f(E^k)$ in a set open in $f(E^k)$. There is a simplicial triangulation of E^k, say K, such that (identifying $|K|$ with E^k) for every simplex $s \in K$ either $f(|s|) \subset Y$ or for some $1 \leq i \leq m$, $f(|s|) \subset e_i - \dot{e}_i$. Let A' be the subpolyhedron of E^k which is the space of all simplexes $s \in K$ such that $f(|s|) \subset Y$, and for $1 \leq i \leq m$ let B_i be the subpolyhedron which is the space of all simplexes s of K such that $f(|s|) \subset e_i - \dot{e}_i$. Then $S^{k-1} \subset A'$, $E^k = A' \cup B_1 \cup \cdots \cup B_m$, and if $i \neq j$, then $B_i - A'$ is disjoint from $B_j - A'$. Let $\dot{B}_i = B_i \cap A'$ and observe that $(B_i,\dot{B}_i)$ is a relative CW complex, with $\dim (B_i - \dot{B}_i) \leq k \leq n - 1$.

For $1 \leq i \leq m$ the pair $((e_i - \dot{e}_i), (e_i - \dot{e}_i) - x_i)$ is homeomorphic to $(E^n - S^{n-1}, (E^n - S^{n-1}) - 0)$ and has the same homotopy groups as (E^n,S^{n-1}). By corollary 7.2.4, (E^n,S^{n-1}) is $(n - 1)$-connected. It follows from theorem 13 that $f \mid (B_i,\dot{B}_i)$ is homotopic relative to $\dot{B}_i$ to a map from B_i to $(e_i - \dot{e}_i) - x_i$. Because $B_i - \dot{B}_i$ is disjoint from $B_j - \dot{B}_j$ for $i \neq j$, these homotopies fit together to define a homotopy relative to A' of f to some map f' such that $f'(E^k) \subset Y$. Clearly, A is a strong deformation retract of Y. Therefore f' is homotopic relative to S^{k-1} to a map f'' such that $f''(E^k) \subset A$. Then $f \simeq f' \simeq f''$, all homotopies relative to S^{k-1}. Therefore (X,A) is $(n - 1)$-connected. ∎

16 COROLLARY *If (X,A) is a relative CW complex, then for any $n \geq 0$, $(X, (X,A)^n)$ is n-connected.*

PROOF We prove by induction on m that $((X,A)^m, (X,A)^n)$ is n-connected for $m > n$. Since $(X,A)^{n+1}$ is obtained from $(X,A)^n$ by adjoining $(n + 1)$-cells, it follows from lemma 15 that $((X,A)^{n+1}, (X,A)^n)$ is n-connected. Assume $m > n + 1$ and that $((X,A)^{m-1}, (X,A)^n)$ is n-connected. By lemma 15, the pair $((X,A)^m, (X,A)^{m-1})$ is $(m - 1)$-connected, and since $n < m - 1$, it is also n-connected. Then $\pi_0((X,A)^n) \to \pi_0((X,A)^{m-1})$ and $\pi_0((X,A)^{m-1}) \to \pi_0((X,A)^m)$ are both surjective, whence $\pi_0((X,A)^n) \to \pi_0((X,A)^m)$ is also surjective. Furthermore, for any $x \in (X,A)^n$, it follows from the exactness of the homotopy sequence of the triple $((X,A)^m, (X,A)^{m-1}, (X,A)^n)$, with base point x, that $\pi_k((X,A)^m, (X,A)^n, x) = 0$ for $1 \leq k \leq n$. By corollary 7.2.2, $((X,A)^m, (X,A)^n)$ is n-connected.

To show that $(X, (X,A)^n)$ is n-connected, if $0 \leq k \leq n$ and $\alpha\colon (E^k,S^{k-1}) \to (X, (X,A)^n)$, then because $\alpha(E^k)$ is compact and X has a topology coherent with the subspaces $(X,A)^m$, there is $m > n$ such that $\alpha(E^k) \subset (X,A)^m$. Hence α can be regarded as a map from (E^k,S^{k-1}) to $((X,A)^m, (X,A)^n)$ for some $m > n$. Because $((X,A)^m, (X,A)^n)$ is n-connected, α is homotopic relative to S^{k-1} to some map of E^k to $(X,A)^n$. ∎

Given relative CW complexes (X,A) and (X',A'), a map $f\colon (X,A) \to (X',A')$ is said to be *cellular* if $f((X,A)^k) \subset (X',A')^k$ for all k. Similarly, a homotopy $F\colon (X,A) \times I \to (X',A')$ is said to be *cellular* if $F((X,A) \times I)^k \subset (X',A')^k$ for all k. Analogous to the simplicial-approximation theorem is the following *cellular-approximation theorem*.

17 THEOREM *Given a map $f\colon (X,A) \to (X',A')$ between relative CW complexes which is cellular on a subcomplex (Y,B) of (X,A), there is a cellular map $g\colon (X,A) \to (X',A')$ homotopic to f relative to Y.*

PROOF It follows from corollary 16, theorem 13, and theorem 12 that there is a sequence of homotopies $H_k\colon (X,A) \times I \to (X',A')$ relative to Y, for $k \geq 0$, such that

(*a*) $H_0(x,0) = f(x)$ for $x \in X$.
(*b*) $H_k(x,1) = H_{k+1}(x,0)$ for $x \in X$.
(*c*) H_k is a homotopy relative to $(X,A)^{k-1}$.
(*d*) $H_k((X,A)^k \times 1) \subset (X',A')^k$.

Then a homotopy $H\colon (X,A) \times I \to (X',A')$ with the desired properties is defined by

$$H(x,t) = H_{k-1}\left(x, \frac{t - (1 - 1/k)}{(1/k) - 1/(k+1)}\right) \qquad 1 - \frac{1}{k} \leq t \leq 1 - \frac{1}{k+1}$$

$$H(x,1) = H_k(x,1) \qquad x \in (X,A)^k \quad \blacksquare$$

18 COROLLARY *Any map between relative CW complexes is homotopic to a cellular map. If two cellular maps between relative CW complexes are homotopic, there is a cellular homotopy between them.* ■

A continuous map $f\colon X \to Y$ is called an *n-equivalence* for $n \geq 1$ if f induces a one-to-one correspondence between the path components of X and of Y and if for every $x \in X$, $f_\#\colon \pi_q(X,x) \to \pi_q(Y,f(x))$ is an isomorphism for $0 < q < n$ and an epimorphism for $q = n$ (the condition concerning the case $q = n$ is sometimes omitted in the definitions occurring in the literature). A map $f\colon X \to Y$ is called a *weak homotopy equivalence* or ∞*-equivalence* if f is an n-equivalence for all $n \geq 1$. The following results are immediate from the definition and from corollary 7.3.15.

19 *A composite of n-equivalences is an n-equivalence.* ■

20 *Any map homotopic to an n-equivalence is an n-equivalence.* ■

21 *A homotopy equivalence is a weak homotopy equivalence.* ■

Let $f\colon X \to Y$ be a map and let Z_f be the mapping cylinder of f. Then $f = r \circ i$, where $r\colon Z_f \to Y$ is a homotopy equivalence. Therefore f is an n-equivalence if and only if $i\colon X \subset Z_f$ is an n-equivalence. It follows from the exactness of the homotopy sequence of (Z_f,X) and from corollary 7.2.2 that i is an n-equivalence if and only if (Z_f,X) is n-connected.

22 THEOREM *Let $f\colon X \to Y$ be an n-equivalence (n finite or infinite) and let (P,Q) be a relative CW complex, with* $\dim (P - Q) \leq n$. *Given maps $g\colon Q \to X$ and $h\colon P \to Y$ such that $h \mid Q = f \circ g$, there exists a map $g'\colon P \to X$ such that $g' \mid Q = g$ and $f \circ g' \simeq h$ relative to Q.*

PROOF Let Z_f be the mapping cylinder of f, with inclusion maps $i\colon X \subset Z_f$

and j: $Y \subset Z_f$, and retraction r: $Z_f \to Y$ a homotopy inverse of j. Then in

$$\begin{array}{ccc} Q & \subset & P \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle j \circ h} \\ X & \xrightarrow{i} & Z_f \end{array}$$

a homotopy $i \circ g \simeq j \circ h \mid Q$ can be found whose composite with r is constant. By theorem 12, there is a map h': $P \to Z_f$ such that $h' \mid Q = i \circ g$ and such that $r \circ h' \simeq r \circ j \circ h$ relative to Q. We regard h' as a map from (P,Q) to (Z_f,X). Since (Z_f,X) is n-connected and $\dim (P - Q) \leq n$, it follows from theorem 13 that h' is homotopic relative to Q to some map g': $P \to X$. Then $g' \mid Q = g$ and

$$f \circ g' = r \circ i \circ g' \simeq r \circ h' \simeq r \circ j \circ h = h$$

all the homotopies being relative to Q. Hence g' has the desired properties. ∎

23 COROLLARY *Let f: $X \to Y$ be an n-equivalence (n finite or infinite) and consider the map*

$$f_{\#}: [P;X] \to [P;Y]$$

If P is a CW complex of dimension $\leq n$, this map is surjective, and if $\dim P \leq n - 1$, *it is injective.*

PROOF The first part follows from theorem 22 applied to the relative CW complex $(P, \varnothing)$.

For the second part, we apply theorem 22 to the relative CW complex $(P \times I, P \times \dot{I})$. Given g_0, g_1: $P \to X$ such that $f \circ g_0 \simeq f \circ g_1$, there is a map g: $P \times \dot{I} \to X$ such that $g(z,0) = g_0(z)$ and $g(z,1) = g_1(z)$ for $z \in P$ and a map h: $P \times I \to Y$ such that $h \mid P \times \dot{I} = f \circ g$. Since $\dim (P \times I) \leq n$, by theorem 22 there is a mapping g': $P \times I \to X$ such that $g' \mid P \times \dot{I} = g$. Then g' is a homotopy from g_0 to g_1, showing that $[g_0] = [g_1]$. ∎

24 COROLLARY *A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

PROOF It follows from statement 21 that a map which is a homotopy equivalence is always a weak homotopy equivalence. Conversely, if f: $X \to Y$ is a weak homotopy equivalence between CW complexes, it follows from corollary 23 that f induces bijections

$$f_{\#}: [Y;X] \to [Y;Y] \qquad f_{\#}: [X;X] \to [X;Y]$$

If g: $Y \to X$ is any map such that $f_{\#}[g] = [1_Y]$, then $f \circ g \simeq 1_Y$, and also

$$f_{\#}[g \circ f] = [f \circ g \circ f] = [1_Y \circ f] = [f \circ 1_X] = f_{\#}[1_X]$$

Therefore $[g \circ f] = [1_X]$ or $g \circ f \simeq 1_X$, and so f is a homotopy equivalence. ∎

Thus, for CW complexes the concepts of homotopy equivalence and weak homotopy equivalence coincide. The following theorem is a direct consequence of the Whitehead theorem 7.5.9.

25 THEOREM *A weak homotopy equivalence induces isomorphisms of the corresponding integral singular homology groups. Conversely, a map between simply connected spaces which induces isomorphisms of the corresponding integral singular homology groups is a weak homotopy equivalence.* ∎

7 HOMOTOPY FUNCTORS

In this section we shall study a general class of functors on the homotopy category of path-connected pointed spaces. The main result characterizes, on the subcategory of CW complexes, those functors of the form π^Y for some Y in terms of simple properties. In the next section we shall apply this result to prove the existence of approximations to any space by a CW complex.[1]

In a category $\mathcal{C}$, given objects A and X and morphisms $f_0: A \to X$ and $f_1: A \to X$, an *equalizer* of f_0 and f_1 is a morphism $j: X \to Z$ such that

(*a*) $j \circ f_0 = j \circ f_1$.
(*b*) If $j': X \to Z'$ is a morphism in $\mathcal{C}$ such that $j' \circ f_0 = j' \circ f_1$, there is a morphism $g: Z \to Z'$ such that $j' = g \circ j$.

Note that it is not asserted in condition (*b*) that g is unique.

We define $\mathcal{C}_0$ to be the homotopy category of path-connected pointed spaces having nondegenerate base points.

1 LEMMA *The category $\mathcal{C}_0$ has equalizers.*

PROOF Let A and X be arbitrary objects of $\mathcal{C}_0$ and let $f_0: A \to X$ and $f_1: A \to X$ be maps preserving base points. Let Z be the space obtained from the topological sum $X \vee (A \times I)$ by identifying $(a,0) \in A \times I$ with $f_0(a) \in X$, $(a,1) \in A \times I$ with $f_1(a) \in X$ for all $a \in A$, and $(a_0,t) \in A \times I$ with $(a_0,0)$ (a_0 the base point of A) for all $t \in I$. Then Z is an object of $\mathcal{C}_0$ and the inclusion map $j: X \subset Z$ has the property that $j \circ f_0 \simeq j \circ f_1$ [in fact, the composite $A \times I \subset X \vee (A \times I) \to Z$ is a homotopy from $j \circ f_0$ to $j \circ f_1$]. Furthermore, if $j': X \to Z'$ is a map such that $j' \circ f_0 \simeq j' \circ f_1$, there is a map $G: X \vee (A \times I) \to Z'$ such that $G \mid X = j'$ and $G \mid A \times I$ is a homotopy from $j' \circ f_0$ to $j' \circ f_1$. Then G is compatible with the collapsing map $k: X \vee (A \times I) \to Z$, so there is a map $g: Z \to Z'$ such that $G = g \circ k$. Then $j' = g \circ j$, and therefore $[j]: X \to Z$ is an equalizer of $[f_0]$ and $[f_1]$ in $\mathcal{C}_0$. ∎

2 LEMMA *Let $\{Y_n\}_{n \geq 0}$ be objects of $\mathcal{C}_0$ that are subspaces of a space Y in $\mathcal{C}_0$ such that $Y_n \subset Y_{n+1}$ is a cofibration for all $n \geq 0$, $Y = \cup_n Y_n$, and Y has the topology coherent with $\{Y_n\}$. Let $i_n: Y_n \subset Y_{n+1}$, $1_n: Y_n \subset Y_n$, and $j_n: Y_n \subset Y$ be the inclusion maps. Then the homotopy class $[\{j_n\}]: \bigvee Y_n \to Y$ is an equalizer in $\mathcal{C}_0$ of the homotopy classes*

[1] The techniques of this section are based on E. Brown, Cohomology theories, *Annals of Mathematics*, vol. 75, pp. 467–484, 1962.

$$[\vee i_n]\colon \vee Y_n \to \vee Y_n \quad \textit{and} \quad [\vee 1_n]\colon \vee Y_n \to \vee Y_n$$

PROOF Since $j_{n+1} \circ i_n = j_n \circ 1_n$, it follows that $\{j_n\} \circ \vee i_n = \{j_n\} \circ \vee 1_n$. Given a map $j'\colon \vee Y_n \to Z'$ such that $j' \circ \vee i_n \simeq j' \circ \vee 1_n$, let $j'_n\colon Y_n \to Z'$ be defined by $j'_n = j' \mid Y_n$. Then $j'_{n+1} \circ i_n \simeq j'_n$, and using the fact that $Y_n \subset Y_{n+1}$ is a cofibration and by induction on n, there is a sequence of maps $g_n\colon Y_n \to Z'$ such that $g_n \simeq j'_n$ and $g_{n+1} \circ i_n = g_n$. Let $g\colon Y \to Z'$ be the map such that $g \mid Y_n = g_n$. If $j = \{j_n\}\colon \vee Y_n \to Y$, then $g \circ j \simeq j'$ completing the proof. ■

A *homotopy functor* is a contravariant functor H from $\mathcal{C}_0$ to the category of pointed sets such that both of the following hold:

(*a*) If $[j]\colon X \to Z$ is an equalizer of $[f_0], [f_1]\colon A \to X$ and if $u \in H(X)$ is such that $H([f_0])u = H([f_1])u$, there is $v \in H(Z)$ such that $H([j])v = u$.
(*b*) If $\{X_\lambda\}_\lambda$ is an indexed family of objects in $\mathcal{C}_0$ and $i_\lambda\colon X_\lambda \subset \vee X_\lambda$, there is an equivalence

$$\{H[i_\lambda]\}_\lambda\colon H(\vee X_\lambda) \approx \times H(X_\lambda)$$

If $f\colon X \to Y$ is a base-point-preserving map and H is a homotopy functor, we shall also use $H(f)$ for $H([f])$. If $X \subset X'$ and $u \in H(X')$, we use $u \mid X$ for $H(i)u$, where $i\colon X \subset X'$.

If X is a one-point space, and X_1 and X_2 are both equal to X, then $X_1 \vee X_2$ is also equal to X, and the equivalence of condition (*b*)

$$\{H(i_1),H(i_2)\}\colon H(X_1 \vee X_2) \approx H(X_1) \times H(X_2)$$

corresponds to the diagonal map of $H(X)$ to $H(X) \times H(X)$. Because this is a bijection, $H(X)$ consists of a single element.

Following are some examples.

3 Let Y be a pointed space. Then the functor π^Y on $\mathcal{C}_0$ defined as in Sec. 1.3 (that is, $\pi^Y(X) = [X;Y]$ for an object X in $\mathcal{C}_0$) is a homotopy functor.

4 Fix an integer $n > 0$ and an abelian group G. Then the functor $H(X) = H^n(X,x_0;\, G)$ (singular cohomology) on $\mathcal{C}_0$ is a homotopy functor called the *nth cohomology functor with coefficients* G.

5 Let G be an arbitrary group (possibly nonabelian). There is a homotopy functor H such that $H(X)$ is the set of all homomorphisms $\pi_1(X,x_0) \to G$ with the trivial homomorphism as base point.

An important result of this section is that on the subcategory of pointed path-connected CW complexes every homotopy functor is naturally equivalent to π^Y for a suitable pointed space Y.

6 LEMMA *Let $\nu\colon SX \to SX \vee SX$ be the comultiplication map. If X is in $\mathcal{C}_0$ and H is a homotopy functor, the composite*

$$H(SX) \times H(SX) \xrightarrow{\{H(i_1),H(i_2)\}^{-1}} H(SX \vee SX) \xrightarrow{H(\nu)} H(SX)$$

is a group multiplication on $H(SX)$, which is abelian if X is a suspension. If H

is a homotopy functor taking values in the category of groups, the two group structures on $H(SX)$ agree.

PROOF Each of the group properties for this multiplication follows from the corresponding H cogroup property of ν. The final statement of the lemma follows from theorem 1.6.8, because the two multiplications in $H(SX)$ are mutually distributive. ■

In particular, for any homotopy functor H, $H(S^q)$ is a group for $q \geq 1$ and abelian for $q \geq 2$ and is called the *qth coefficient group of H.* Thus the qth coefficient group of the functor π^Y of example 3 is $\pi_q(Y)$. The qth coefficient group of the nth cohomology functor with coefficients G of example 4 is 0 if $q \neq n$ and isomorphic to G if $q = n$. The qth coefficient group of the functor of example 5 is G if $q = 1$ and 0 if $q > 1$.

If Y is an object of $\mathcal{C}_0$ and H is a homotopy functor, any element $u \in H(Y)$ determines a natural transformation

$$T_u\colon \pi^Y \to H$$

defined by $T_u([f]) = H([f])(u)$ for $[f] \in [X;Y]$. For a suspension SX, T_u is a homomorphism from $\pi^Y(SX) = [SX;Y]$ to the group $H(SX)$, with the multiplication of lemma 6 (because both group multiplications are induced by the comultiplication $\nu\colon SX \to SX \vee SX$). An element $u \in H(Y)$ is said to be *n-universal for H*, where $n \geq 1$, if the homomorphism

$$T_u\colon \pi^Y(S^q) \to H(S^q)$$

is an isomorphism for $1 \leq q < n$ and an epimorphism for $q = n$. An element $u \in H(Y)$ is said to be *universal* for H if it is n-universal for all $n \geq 1$, in which case Y is called a *classifying space for H.*

7 THEOREM *Assume that H is a homotopy functor with universal elements $u \in H(Y)$ and $u' \in H(Y')$ and let $f\colon Y \to Y'$ be a map such that $H(f)u' = u$. Then f is a weak homotopy equivalence.*

PROOF Since Y and Y' are path connected, this is a consequence of the commutativity of the diagram (for $q \geq 1$)

$$\begin{tikzcd}[S^q;Y] \arrow[rr, "f_\#"] \arrow[dr, "T_u"', "\approx"] & & {[S^q;Y']} \arrow[dl, "\approx"', "T_{u'}"] \\ & H(S^q) & \end{tikzcd}$$ ■

The same kind of argument establishes the next result.

8 LEMMA *Let Y be an object of $\mathcal{C}_0$ and let Y' be an arbitrary path-connected space. A map $f\colon Y \to Y'$ is a weak homotopy equivalence if and only if $[f] \in [Y;Y'] = \pi^{Y'}(Y)$ is universal for $\pi^{Y'}$.* ■

We are heading toward a proof of the existence of universal elements for any homotopy functor. The following two lemmas will be used in this proof.

9 LEMMA *Let H be a homotopy functor, Y an object in $\mathcal{C}_0$, and $u \in H(Y)$. There exist an object Y' in $\mathcal{C}_0$, obtained from Y by attaching 1-cells, and a 1-universal element $u' \in H(Y')$ such that $u' \mid Y = u$.*

PROOF For each $\lambda \in H(S^1)$ let $S_\lambda{}^1$ be a 1-sphere and define $Y' = Y \vee \bigvee_\lambda S_\lambda{}^1$. Then Y' is an object of $\mathcal{C}_0$ obtained from Y by attaching 1-cells. If g_λ is the composite $S^1 \approx S_\lambda{}^1 \subset Y'$, it follows from condition (*b*) on page 407 that there is an element $u' \in H(Y')$ such that $u' \mid Y = u$ and $H(g_\lambda)u' = \lambda$ for $\lambda \in H(S^1)$. Since $T_{u'}([g_\lambda]) = \lambda$, $T_{u'}([S^1;Y']) = H(S^1)$, and u' is 1-universal. ∎

10 LEMMA *Let H be a homotopy functor and $u \in H(Y)$ an n-universal element for H, with $n \geq 1$. There exist an object Y' in $\mathcal{C}_0$, obtained from Y by attaching $(n+1)$-cells, and an $(n+1)$-universal element $u' \in H(Y')$ such that $u' \mid Y = u$.*

PROOF For each $\lambda \in H(S^{n+1})$ let $S_\lambda{}^{n+1}$ be an $(n+1)$-sphere, and for each map $\alpha\colon S^n \to Y$ such that $H(\alpha)u = 0$ attach an $(n+1)$-cell $e_\alpha{}^{n+1}$ to Y by α. Let Y' be the space obtained from $Y \vee \bigvee_\lambda S_\lambda{}^{n+1}$ by attaching the $(n+1)$-cells $\{e_\alpha{}^{n+1}\}$. Then Y' is an object of $\mathcal{C}_0$ obtained from Y by attaching $(n+1)$-cells. If $g_\lambda\colon S^{n+1} \to Y \vee \bigvee_\lambda S_\lambda{}^{n+1}$ is the composite $S^{n+1} \approx S_\lambda{}^{n+1} \subset Y \vee \bigvee_\lambda S_\lambda{}^{n+1}$, it follows from condition (*b*) on page 407 that there is an element $\bar{u} \in H(Y \vee \bigvee_\lambda S_\lambda{}^{n+1})$ such that $\bar{u} \mid Y = u$ and $H(g_\lambda)\bar{u} = \lambda$ for $\lambda \in H(S^{n+1})$.

For each map $\alpha\colon S^n \to Y$ such that $H(\alpha)u = 0$ let $S_\alpha{}^n$ be an n-sphere and let $f_0\colon \bigvee_\alpha S_\alpha{}^n \to Y \vee \bigvee_\lambda S_\lambda{}^{n+1}$ be the constant map and let $f_1\colon \bigvee_\alpha S_\alpha{}^n \to Y \vee \bigvee_\lambda S_\lambda{}^{n+1}$ be the map such that $S_\alpha{}^n$ is mapped by α. Then

$$j\colon Y \vee \bigvee_\lambda S_\lambda{}^{n+1} \subset Y'$$

is a map such that $[j]$ is an equalizer of $[f_0]$ and $[f_1]$. Since $H(f_0)\bar{u} = 0 = H(f_1)\bar{u}$, by condition (*a*) on page 407 there is an element $u' \in H(Y')$ such that $H(j)u' = \bar{u}$. Then $u' \mid Y = u$ and to complete the proof we need only show that u' is $(n+1)$-universal.

There is a commutative diagram

$$\begin{tikzcd}
\pi_{q+1}(Y',Y) \arrow[r, "\partial"] & \pi_q(Y) \arrow[r, "i_\#"] \arrow[dr, "T_u"'] & \pi_q(Y') \arrow[r] \arrow[d, "T_{u'}"] & \pi_q(Y',Y) \\
& & H(S^q) &
\end{tikzcd}$$

with the top row exact. Since Y' is obtained from Y by attaching $(n+1)$-cells, it follows from lemma 7.6.15 that $\pi_q(Y',Y) = 0$ for $q \leq n$. Therefore $i_\#$ is an isomorphism for $q < n$ and an epimorphism for $q = n$. Since u is n-universal, T_u is an isomorphism for $q < n$ and an epimorphism for $q = n$. It follows that $T_{u'}$ is also an isomorphism for $q < n$ and an epimorphism for $q = n$. Furthermore, if $a \in [S^n;Y]$ is in the kernel of T_u, then a is represented by a map $\alpha\colon S^n \to Y$ and

$$a = [\alpha] \in \partial(\pi_{n+1}(e_\alpha{}^{n+1},\dot{e}_\alpha{}^{n+1})) \subset \partial(\pi_{n+1}(Y',Y)) = \ker i_\#$$

Therefore, for $q = n$, $\ker T_u = \ker i_\#$, and so $T_{u'}$ is an isomorphism from $\pi_n(Y')$ to $H(S^n)$. For any $\lambda \in H(S^{n+1})$ the map $j \circ g_\lambda \colon S^{n+1} \to Y'$ has the property that

$$T_{u'}([j \circ g_\lambda]) = H(g_\lambda)\bar{u} = \lambda$$

showing that $T_{u'}$ is an epimorphism for $q = n + 1$, and so u' is $(n + 1)$-universal. ■

11 THEOREM *Let H be a homotopy functor, let Y be an object in $\mathcal{C}_0$, and let $u \in H(Y)$. Then there are a classifying space Y' for H containing Y such that (Y',Y) is a relative CW complex and a universal element $u' \in H(Y')$ such that $u' \mid Y = u$.*

PROOF Using lemmas 9 and 10, we construct, by induction on n, a sequence of objects $\{Y_n\}_{n \geq 0}$ in $\mathcal{C}_0$ and elements $u_n \in H(Y_n)$ such that

(*a*) $Y_0 = Y$ and $u_0 = u$.
(*b*) Y_{n+1} is obtained from Y_n by attaching $(n + 1)$-cells, where $n \geq 0$.
(*c*) $u_{n+1} \mid Y_n = u_n$.
(*d*) u_n is n-universal for $n \geq 1$.

It follows from (*b*) above that $Y' = \bigcup Y_n$ topologized with the topology coherent with $\{Y_n\}$ is a path-connected pointed space containing Y such that (Y',Y) is a relative CW complex. By lemma 2, the homotopy class $[\{j_n\}] \colon \bigvee Y_n \to Y'$ is an equalizer of the homotopy classes $[\bigvee i_n] \colon \bigvee Y_n \to \bigvee Y_n$ and $[\bigvee 1_n] \colon \bigvee Y_n \to \bigvee Y_n$. By condition (*b*) on page 407 there is an element $\bar{u} \in H(\bigvee Y_n)$ such that $\bar{u} \mid Y_n = u_n$. It follows from (*c*) above that $H(\bigvee i_n)\bar{u} = H(\bigvee 1_n)\bar{u}$, and by condition (*a*) on page 407 there is an element $u' \in H(Y')$ such that $H(\{j_n\})u' = \bar{u}$ (that is, $u' \mid Y_n = u_n$ for $n \geq 0$). Then $u' \mid Y = u$, and it remains to show that u' is universal.

By the definition of Y' and u', there is a commutative diagram for $q \geq 1$

$$\begin{tikzcd} \varinjlim \{\pi_q(Y_n)\} \arrow[rr, "\approx"] \arrow[dr, "\{T_{u_n}\}"'] & & \pi_q(Y') \arrow[dl, "T_{u'}"] \\ & H(S^q) & \end{tikzcd}$$

Since u_n is n-universal, T_{u_n} is an isomorphism for $n > q$, and so the left-hand map is an isomorphism. Therefore $T_{u'}$ is also an isomorphism, and u' is universal. ■

12 COROLLARY *For any homotopy functor there exist classifying spaces which are CW complexes.*

PROOF Apply theorem 11 to a one-point space Y, with u the unique element of $H(Y)$. ■

13 COROLLARY *Let $u \in H(Y)$ be a universal element for a homotopy functor H. Let (X,A) be a relative CW complex, where A and X are objects* *in $\mathcal{C}_0$. Given a map $g: A \to Y$ and an element $v \in H(X)$ such that $v \mid A = H(g)u$, there exists a map $g': X \to Y$ such that $g = g' \mid A$ and $v = H(g')u$.*

PROOF Let $i: X \subset X \vee Y$ and $i': Y \subset X \vee Y$ and let $j: X \vee Y \to Z$ be a map such that $[j]$ is an equalizer of $[i \circ f]$ (where $f: A \subset X$) and $[i' \circ g]$. By condition (b) on page 407, there is an element $\bar{v} \in H(X \vee Y)$ such that $\bar{v} \mid X = v$ and $\bar{v} \mid Y = u$. Since $H(f)v = H(g)u$, it follows that $H(i \circ f)\bar{v} = H(i' \circ g)\bar{v}$, and by condition (a) on page 407, there is an element $\bar{u} \in H(Z)$ such that $H(j)\bar{u} = \bar{v}$. We now apply theorem 11 to $\bar{u}$ to obtain a Y' containing Z and a universal element $u' \in H(Y')$ such that $\bar{u} = u' \mid Z$. Let $j': Y \to Y'$ be the composite

$$Y \overset{i'}{\subset} X \vee Y \xrightarrow{j} Z \overset{h}{\subset} Y'$$

Then $H(j')u' = u$, and by theorem 7, j' is a weak homotopy equivalence. Since the composite

$$A \overset{f}{\subset} X \overset{i}{\subset} X \vee Y \xrightarrow{j} Z \overset{h}{\subset} Y'$$

is homotopic to $j' \circ g$, it follows from the fact that f is a cofibration that there is a map $\bar{g}: X \to Y'$ such that $\bar{g} \mid A = j' \circ g$ and $\bar{g}$ is homotopic to $h \circ j \circ i$. Since j' is a weak homotopy equivalence, by theorem 7.6.22, there is a map $g': X \to Y$ such that $g' \mid A = g$ and $j' \circ g' \simeq \bar{g}$. Then

$$H(g')u = H(g')H(j')u' = H(i)H(j)H(h)u' = \bar{v} \mid X = v$$

showing that g' has the requisite properties. ■

14 THEOREM *If Y is a classifying space and $u \in H(Y)$ is a universal element for a homotopy functor H, then for any CW complex X in $\mathcal{C}_0$, T_u is a natural equivalence of $\pi^Y(X)$ with $H(X)$.*

PROOF Given $v \in H(X)$, apply corollary 13, with $A = x_0$ and g the constant map, to obtain a map $g': X \to Y$ such that $H(g')u = v$. Then $T_u[g'] = v$, proving that T_u is surjective.

If $g_0, g_1: X \to Y$ are maps such that $T_u[g_0] = T_u[g_1]$, let X' be the CW complex $X \times I/x_0 \times I$, with $(X')^q = [(X^q \times \dot{I}) \cup (X^{q-1} \times I)]/(x_0 \times I)$ for $q \geq 0$. Let $v \in H(X')$ be defined by $v = H(h)H(g_0)u$, where $h: X' \to X$ is the map $h([x,t]) = x$. Let $A = X \times \dot{I}/x_0 \times \dot{I}$ and let $g: A \to Y$ be the map such that $g([x,0]) = g_0(x)$ and $g([x,1]) = g_1(x)$. Then $H(g)u = v \mid A$, and by corollary 13, there is a map $g': X' \to Y$ such that $g' \mid A = g$. Then the composite

$$X \times I \to X \times I/x_0 \times I \xrightarrow{g'} Y$$

is a homotopy relative to x_0 from g_0 to g_1, showing that T_u is injective. ■

15 COROLLARY *If Y and Y' are classifying spaces which are CW complexes and $u \in H(Y)$ and $u' \in H(Y')$ are universal elements for a homotopy functor H, there is a homotopy equivalence $h: Y \to Y'$, unique up to homotopy, such that $H(h)u' = u$.*

PROOF By theorem 14, there exists a unique homotopy class $[g]: Y \to Y'$ such that $H(g)u' = u$. By theorem 7, g is a weak homotopy equivalence. By corollary 7.6.24, g is a homotopy equivalence. ∎

8 WEAK HOMOTOPY TYPE

In this section we shall show that any space can be approximated by CW complexes. This leads to an equivalence relation based on weak homotopy equivalence which is weaker than homotopy equivalence. We shall also consider the same equivalence relation in the category of maps. This will be used in defining and analyzing the general relative-lifting problem.

A *relative CW approximation* to a pair (X,A) consists of a relative CW complex (Y,A) and a weak homotopy equivalence $f: Y \to X$ such that $f(a) = a$ for all $a \in A$. A *CW approximation* to a space X is a relative CW approximation to $(X, \varnothing)$.

1 THEOREM *Any pair has relative CW approximations, and two relative CW approximations to the same pair have the same homotopy type.*

PROOF First we consider the case where X is path connected. Let $x_0 \in X$ and let $\{A_j\}_{j \in J}$ be the set of path components of A, and for each $j \in J$ choose a point $a_j \in A_j$. There is a relative CW complex (A',A) with $(A',A)^0 = A \cup e^0$, where e^0 is a single point and

$$A' = (A',A)^1 = (A',A)^0 \cup \bigcup_{j \in J} e_j^1$$

where e_j^1 is a 1-cell such that $\dot{e}_j^1 = e^0 \cup a_j$ for $j \in J$. Let $g: A' \to X$ be a map such that $g(a) = a$ for $a \in A$, $g(e^0) = x_0$, and $g \mid e_j^1$ is a path in X with end points x_0 and a_j for each $j \in J$. Then A' is a path-connected space with nondegenerate base point e^0 and $[g] \in \pi^X(A')$. It follows from theorem 7.7.11 that there is a relative CW complex (Y,A') [which can be chosen such that $(Y,A')^1 = A' \vee \bigvee S_\alpha^1$] and a universal element $[g'] \in \pi^X(Y)$ for π^X such that $g' \mid A' \simeq g$. Since $A' \subset Y$ is a cofibration, there is a map $f: Y \to X$ such that $[f] \in \pi^X(Y)$ is universal for π^X and $f \mid A' = g$. By lemma 7.7.8, f is a weak homotopy equivalence. Since (Y,A) is a relative CW complex [with $(Y,A)^0 = (A',A)^0$ and $(Y,A)^q = (Y,A')^q$ for $q \geq 1$] and since $f(a) = a$ for $a \in A$, (Y,A) and f constitute a relative CW approximation to (X,A).

Next we consider the case where X is not path connected and we let $\{X_\alpha\}$ be the set of path components of X. By the case already considered, for each α there is a relative CW approximation $f_\alpha: (Y_\alpha, X_\alpha \cap A) \to (X_\alpha, X_\alpha \cap A)$. Let Y be the space obtained from the disjoint union $A \cup \bigcup Y_\alpha$ by identifying $x \in X_\alpha \cap A \subset Y_\alpha$ with $x \in A$ for each α and let $k: A \cup \bigcup Y_\alpha \to Y$ be the collapsing map. Then $k \mid A: A \to Y$ is an imbedding and (Y,A) is a relative CW complex with $(Y,A)^q = k(A \cup \bigcup (Y_\alpha, X_\alpha \cap A)^q)$ for all $q \geq 0$. There is a map $f: Y \to X$ such that $fk(a) = a$ for $a \in A$ and $f \circ (k \mid Y_\alpha) = f_\alpha$ for all α.

Since $\{k(Y_\alpha)\}$ is the set of path components of Y and f induces a weak homotopy equivalence of each of these with the corresponding path component X_α of X, f is a weak homotopy equivalence from Y to X. Identifying A with $k(A)$, we see that (Y,A) and f constitute a CW approximation to (X,A).

Given two relative CW approximations to (X,A), say f_1: $(Y_1,A) \to (X,A)$ and f_2: $(Y_2,A) \to (X,A)$, it follows from theorem 7.6.22 that there are maps g_1: $(Y_1,A) \to (Y_2,A)$ and g_2: $(Y_2,A) \to (Y_1,A)$ such that $f_2 \circ g_1 \simeq f_1$ and $f_1 \circ g_2 \simeq f_2$, both homotopies relative to A. Then $f_2 \circ (g_1 \circ g_2) \simeq f_2 \circ 1$ rel A, and by theorem 7.6.22 again, $g_1 \circ g_2 \simeq 1$ rel A. Similarly, $g_2 \circ g_1 \simeq 1$ rel A, and so (Y_1,A) and (Y_2,A) have the same homotopy type. ■

Two spaces X_1 and X_2 will be said to have the same *weak homotopy type* if there exists a space Y and weak homotopy equivalences f_1: $Y \to X_1$ and f_2: $Y \to X_2$. By replacing such a space Y with a CW approximation to it, we see that X_1 and X_2 have the same weak homotopy type if and only if they have CW approximations by the same CW complex.

2 LEMMA *The relation of having the same weak homotopy type is an equivalence relation.*

PROOF The relation is reflexive and symmetric by its definition. To prove it transitive, let X_1, X_2, and X_3 be spaces and let Y_1 and Y_2 be CW complexes such that there exist weak homotopy equivalences

$$\begin{tikzcd} & Y_1 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & Y_2 \arrow[dl, "g_2"'] \arrow[dr, "g_3"] & \\ X_1 & & X_2 & & X_3 \end{tikzcd}$$

Then f_2: $Y_1 \to X_2$ and g_2: $Y_2 \to X_2$ are both CW approximations to X_2, and by theorem 1, there is a homotopy equivalence h: $Y_1 \to Y_2$ such that $f_2 \simeq g_2 \circ h$. Then $g_3 \circ h$: $Y_1 \to X_3$, being the composite of weak homotopy equivalences, is a weak homotopy equivalence. Therefore X_1 and X_3 have the same weak homotopy type. ■

We are interested in applying these ideas to weak fibrations. The main result is that any two fibers of a weak fibration with path-connected base space have the same weak homotopy type.

3 LEMMA *Let p: $E \to B$ be a weak fibration with contractible base space B. For any $b_0 \in B$ the inclusion map i: $p^{-1}(b_0) \subset E$ is a weak homotopy equivalence.*

PROOF Let $F = p^{-1}(b_0)$. Since B is contractible, $\pi_q(B,b_0) = 0$ for $q \geq 0$. From the exactness of the homotopy sequence of p, it follows that for any $e \in F$, i induces an isomorphism $i_\#$: $\pi_q(F,e) \approx \pi_q(E,e)$ for $q \geq 1$ and $i_\#(\pi_0(F,e)) = \pi_0(E,e)$.

It only remains to verify that $i_\#$ maps $\pi_0(F,e)$ injectively into $\pi_0(E,e)$. Assume that e, $e' \in F$ are such that there is a path ω in E from e to e'. Since B is simply connected and $p \circ \omega$ is a closed path in B at b_0, there is a map

$H: I \times I \to B$ such that $H(t,0) = p\omega(t)$ and $H(0,t') = H(1,t') = H(t,1) = b_0$. Let $g: I \times 0 \cup \dot{I} \times I \to E$ be the map defined by $g(t,0) = \omega(t)$, $g(0,t') = e$, and $g(1,t') = e'$. By lemma 7.2.5, there is a map $G: I \times I \to E$ such that $p \circ G = H$ and $G \mid I \times 0 \cup \dot{I} \times I = g$. Let $\omega': I \to E$ be the path defined by $\omega'(t) = G(1,t)$. Then ω' is a path in F from e to e' [because $p\omega'(t) = b_0$], showing that $i_\#: \pi_0(F,e) \to \pi_0(E,e)$ is injective. ∎

4 COROLLARY *Let $p: E \to B$ be a weak fibration and let ω be a path in B. Then $p^{-1}(\omega(0))$ and $p^{-1}(\omega(1))$ have the same weak homotopy type.*

PROOF Let $p': E' \to I$ be the weak fibration induced from p by $\omega: I \to B$. Then $p^{-1}(\omega(0))$ and $p^{-1}(\omega(1))$ are homeomorphic to $p'^{-1}(0)$ and $p'^{-1}(1)$, respectively. By lemma 3, each of the inclusion maps $p'^{-1}(0) \subset E'$ and $p'^{-1}(1) \subset E'$ is a weak homotopy equivalence. The corollary follows from this and lemma 2. ∎

This result implies the following analogue of corollary 2.8.13 for weak fibrations.

5 COROLLARY *If $p: E \to B$ is a weak fibration with path-connected base space, any two fibers have the same weak homotopy type.* ∎

We now consider the category whose objects are continuous maps $\alpha: P'' \to P'$ between topological spaces and whose morphisms (also called *map pairs*) $f: \alpha \to \beta$ are commutative squares

$$\begin{array}{ccc} P'' & \xrightarrow{f''} & Q'' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ P' & \xrightarrow{f'} & Q' \end{array}$$

In this category a *homotopy pair* $H: f_0 \simeq f_1$, where $f_0, f_1: \alpha \to \beta$, is a commutative square

$$\begin{array}{ccc} P'' \times I & \xrightarrow{H''} & Q'' \\ {\scriptstyle\alpha \times 1}\downarrow & & \downarrow{\scriptstyle\beta} \\ P' \times I & \xrightarrow{H'} & Q' \end{array}$$

such that $H'': f_0'' \simeq f_1''$ and $H': f_0' \simeq f_1'$ (note that H is a map pair from $\alpha \times 1_I$ to β). If such a homotopy pair exists, f_0 is said to be *homotopic* to f_1. This is an equivalence relation in the set of map pairs from α to β, and the corresponding equivalence classes are called *homotopy classes*. We use $[\alpha;\beta]$ to denote the set of homotopy classes of map pairs from α to β, and if $f: \alpha \to \beta$ is a map pair, its homotopy class is denoted by $[f]$. It is trivial to verify that the composites of homotopic map pairs are homotopic, so there is a *homotopy category of maps* whose objects are maps $\alpha: P'' \to P'$ and whose morphisms $\alpha \to \beta$ are homotopy classes $[f]$, where $f: \alpha \to \beta$ is a map pair. A map pair $f: \alpha \to \beta$ is called a *homotopy equivalence from α to β* if $[f]$ is an equivalence in the homotopy category of maps. Two maps α and β are said to have the *same homotopy type* if they are equivalent in the homotopy category of maps.

Given a map pair $g: \alpha' \to \alpha$ (or a map pair $h: \beta \to \beta'$) there is an induced map $g^\#: [\alpha;\beta] \to [\alpha';\beta]$ (or $h_\#: [\alpha;\beta] \to [\alpha;\beta']$) such that $g^\#[f] = [f \circ g]$ (or $h_\#[f] = [h \circ f]$). Since $g^\# \circ h_\# = h_\# \circ g^\#$, the function which assigns $[\alpha;\beta]$ to α and β and $g^\#$ and $h_\#$ to $[g]$ and $[h]$, respectively, is a functor of two variables from the product of the homotopy category of maps by itself to the category of sets that is contravariant in α and covariant in β.

If $\alpha: P'' \to P'$ and $\beta: Q'' \to Q'$ are maps, given a map $\bar f$: P' $\to Q''$, there is a map pair $\rho(\bar f): \alpha \to \beta$ consisting of the commutative square

$$\begin{array}{ccc} P'' & \xrightarrow{\bar f \circ \alpha} & Q'' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ P' & \xrightarrow{\beta \circ \bar f} & Q' \end{array}$$

[that is, $(\rho(\bar f))'' = \bar f \circ \alpha$ and $(\rho(\bar f))' = \beta \circ \bar f$]. Given a map pair $f: \alpha \to \beta$, a *lifting* of f is a map $\bar f: P' \to Q''$ such that $\rho(\bar f) = f$. Two liftings $\bar f_0, \bar f_1: P' \to Q''$ of $f: \alpha \to \beta$ are *homotopic relative to* f if there is a homotopy $\bar H: P' \times I \to Q''$ from $\bar f_0$ to $\bar f_1$ such that $\bar H \circ (\alpha \times 1_I)$ and $\beta \circ \bar H$ are both constant homotopies [that is, $\rho(\bar H)$ is the constant homotopy pair from f to f]. Such a map $\bar H$ is called a *homotopy relative to* f, and we write $\bar H: \bar f_0 \simeq \bar f_1 \text{ rel } f$. Homotopy relative to f is an equivalence relation in the set of liftings of f, and the set of equivalence classes is denoted by $[P';Q'']_f$. The *relative-lifting problem* is the study of $[P';Q'']_f$ (for example, do liftings of f exist, and if so, how many homotopy classes relative to f of liftings of f are there?).

6 EXAMPLE If P'' is empty, then a map pair $f: \alpha \to \beta$ consists of a map $f': P' \to Q'$, and a lifting $\bar f: P' \to Q''$ of f is a lifting of f' to Q'' in the sense defined in Sec. 2.2. In this case, if β is a fibration, two liftings $\bar f_0, \bar f_1: P' \to Q''$ of f' are homotopic relative to f if and only if they are fiber homotopic in the sense of Sec. 2.8. Thus the absolute-lifting problem is a special case of a relative-lifting problem.

7 EXAMPLE If α is an inclusion map and Q' is a one-point space, then a map pair $f: \alpha \to \beta$ corresponds bijectively to a map $f'': P'' \to Q''$ and a lifting $\bar f: P' \to Q''$ of f corresponds bijectively to an extension of f'' to P'. In this case two extensions $\bar f_0, \bar f_1: P' \to Q''$ are homotopic relative to f (as liftings) if and only if they are homotopic relative to P''. Thus the extension problem is a special case of a relative-lifting problem.

8 EXAMPLE Let $\bar f_0, \bar f_1: P' \to Q''$ be liftings of a map pair $f: \alpha \to \beta$. Let $R' = P' \times I$ and let R'' be the quotient space of the disjoint union of $P' \times \dot I$ and $P'' \times I$ by the identifications $(z'',0) \in P'' \times I$ equals $(\alpha(z''),0) \in P' \times \dot I$ and $(z'',1) \in P'' \times I$ equals $(\alpha(z''),1)$. Define a map $\gamma: R'' \to R'$ by $\gamma(z'',t) = (\alpha(z''),t)$ for $(z'',t) \in P'' \times I$ and $\gamma(z',t) = (z',t)$ for $(z',t) \in P' \times \dot I$. There is a map pair $g: \gamma \to \beta$ consisting of the maps $g'': R'' \to Q''$ and $g': R' \to Q'$ such that

$g''(z'',t) = f''(z'')$ for $(z'',t) \in P'' \times I$, $g''(z',0) = \bar{f}_0(z')$ and $g''(z',1) = \bar{f}_1(z')$ for $z' \in P'$, and $g'(z',t) = f'(z')$ for $(z',t) \in P' \times I$. Then $\bar{f}_0$ and $\bar{f}_1$ are homotopic relative to f if and only if there exists a lifting of g.

We are particularly interested in the relative-lifting problem in case α is the inclusion map of a relative CW complex and β is a weak fibration. Thus, if $i: A \subset X$ is an inclusion map and $p: E \to B$ is a weak fibration, a map pair $f: i \to p$ consists of a map $f': X \to B$ and a lifting $f'': A \to E$ of $f' \mid A$. A lifting $\bar{f}$ of f is a lifting of f' to E, which is an extension of f''. Two liftings of f are homotopic relative to f if and only if there is a fiber homotopy relative to A between them. The following relative homotopy extension theorem is the main reason for giving particular attention to this case.

9 THEOREM *Let (X,A) be a relative CW complex, with inclusion map $i: A \subset X$, and let $p: E \to B$ be a weak fibration. Given a map $\bar{f}: X \to E$ and a homotopy pair $H: i \times 1_I \to p$ consisting of a homotopy $H': X \times I \to B$ starting at $p \circ \bar{f}$ and a homotopy $H'': A \times I \to E$ starting at $\bar{f} \circ i$, there is a homotopy $\bar{H}: X \times I \to E$ starting at $\bar{f}$ such that $H' = p \circ \bar{H}$ and $H'' = \bar{H} \circ (i \times 1_I)$.*

PROOF Let $g: X \times 0 \cup A \times I \to E$ be the map defined by $g(x,0) = \bar{f}(x)$ for $x \in X$ and $g(a,t) = H''(a,t)$ for $a \in A$ and $t \in I$. Then H' is an extension of $p \circ g$, and by the standard stepwise-extension procedure over the successive skeleta of (X,A) (applied to polyhedral pairs in the proof of theorem 7.2.6 and equally applicable to any relative CW complex), there is a map $\bar{H}: X \times I \to E$ such that $p \circ \bar{H} = H'$ and $\bar{H} \mid X \times 0 \cup A \times I = g$. Then $\bar{H}$ has the desired properties. ∎

Let us reinterpret this last result. A map pair $f: i \to p$ is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f''} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f'} & B \end{array}$$

Therefore, if we let $B^X \times' E^A$ denote the fibered product of the map $B^X \to B^A$ induced by restriction and the map $E^A \to B^A$ induced by p, the pair (f',f'') is a point of $B^X \times' E^A$. In this way the set of map pairs $f: i \to p$ is identified with the fibered product $B^X \times' E^A$. The map ρ corresponds to a map $\rho: E^X \to B^X \times' E^A$, and $[X;E]_f$ is the set of path components of $\rho^{-1}(f)$.

10 COROLLARY *Let (X,A) be a relative CW complex with X locally compact Hausdorff, with inclusion map $i: A \subset X$, and let $p: E \to B$ be a weak fibration. Then $\rho: E^X \to B^X \times' E^A$ is a weak fibration.*

PROOF Given a map $g: I^n \to E^X$ and a homotopy $H: I^n \times I \to B^X \times' E^A$ starting with $\rho(g)$, the exponential correspondence assigns to g a map $\bar{g}: X \times I^n \to E$ and to H a homotopy pair H_1 from $(i \times 1_{I^n}) \times 1_I$ to p, starting with $\rho(\bar{g})$. By theorem 9, there is a homotopy $\bar{H}_1: X \times I^n \times I \to E$ starting with $\bar{g}$ such that $\rho(\bar{H}_1) = H_1$. Then the exponential correspondence associates to $\bar{H}_1$ a map $G: I^n \times I \to E^X$ starting with g such that $\rho \circ G = H$. ■

It follows from corollaries 10 and 4 that if $f_0, f_1: i \to p$ are homotopic map pairs with X locally compact Hausdorff, then $[X;E]_{f_0}$ and $[X;E]_{f_1}$ are in one-to-one correspondence. Thus the relative-lifting problem for f_0 is equivalent to the relative-lifting problem for f_1.

Given weak fibrations $p_1: E_1 \to B_1$ and $p_2: E_2 \to B_2$, a map pair $g: p_1 \to p_2$ is called a *weak homotopy equivalence* if $g'': E_1 \to E_2$ and $g': B_1 \to B_2$ are weak homotopy equivalences. We shall show that a weak homotopy equivalence in the category of maps has much the same properties as a weak homotopy equivalence in the category of spaces. The following analogue of theorem 7.6.22 is our starting point.

11 LEMMA *Let (X,A) be a relative CW complex, with inclusion map $i: A \subset X$, and let $g: p_1 \to p_2$ be a weak homotopy equivalence between weak fibrations. Given a map pair $f: i \to p_1$ and a lifting $\bar{h}: X \to E_2$ of the map pair $g \circ f$, there is a lifting $\bar{f}: X \to E_1$ of f such that $g'' \circ \bar{f}$ and $\bar{h}$ are homotopic relative to $g \circ f$.*

PROOF The proof involves two applications of theorem 7.6.22 and then two applications of theorem 9. We shall not make specific reference to these when they are invoked.

We have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f''} & E_1 & \xrightarrow{g''} & E_2 \\ {\scriptstyle i}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ X & \xrightarrow{f'} & B_1 & \xrightarrow{g'} & B_2 \end{array}$$

in which g'' and g' are weak homotopy equivalences, and we are given a map $\bar{h}: X \to E_2$ such that $\bar{h} \circ i = g'' \circ f''$ and $p_2 \circ \bar{h} = g' \circ f'$. Then there is a map $\bar{\bar{f}}: X \to E_1$ such that $\bar{\bar{f}} \circ i = f''$ and a homotopy $G'': g'' \circ \bar{\bar{f}} \simeq \bar{h}$ rel A. The maps $p_1 \circ \bar{\bar{f}}$ and f' agree on A and $p_2 \circ G''$ is a homotopy relative to A from $g' \circ p_1 \circ \bar{\bar{f}} = p_2 \circ g'' \circ \bar{\bar{f}}$ to $g' \circ f' = p_2 \circ \bar{h}$. Therefore there is a homotopy $F': p_1 \circ \bar{\bar{f}} \simeq f'$ rel A and a homotopy $H': g' \circ F' \simeq p_2 \circ G''$ rel $A \times I \cup X \times \dot{I}$.

Let $F'': X \times I \to E_1$ be a lifting of F' such that $F''(x,0) = \bar{\bar{f}}(x)$ for $x \in X$ and $F''(a,t) = f''(a)$ for $a \in A$ and $t \in I$. Define $\bar{f}: X \to E_1$ by $\bar{f}(x) = F''(x,1)$. We show that $\bar{f}$ has the desired properties. It is clearly a lifting of f.

The maps $g'' \circ F''$ and G'' are homotopies relative to A from $g'' \circ \bar{\bar{f}}$ to $g'' \circ \bar{f}$ and to $\bar{h}$, respectively, and H' is a homotopy from $p_2 \circ g'' \circ F''$ to $p_2 \circ G''$ rel $A \times I \cup X \times \dot{I}$. Since there is a homeomorphism of $(X \times I \times I, A \times I \times I)$ onto itself taking $X \times (I \times \dot{I} \cup 0 \times I)$ onto $X \times I \times 0$, there is a lifting H'' of H' which is a homotopy from $g'' \circ F''$ to G'' rel $X \times 0 \cup A \times I$. Then the map $H: X \times I \to E_2$ defined by $H(x,t) = H''(x,1,t)$ is a homotopy from $g'' \circ \bar{f}$ to h relative to $g \circ f$. ■

This gives us the following important result.

12 THEOREM *Let (X,A) be a relative CW complex, with inclusion map $i\colon A \subset X$, and let $g\colon p_1 \to p_2$ be a weak homotopy equivalence between weak fibrations. Given a map pair $f\colon i \to p_1$, the map pair g induces a bijection*

$$g''_{\#}\colon [X;E_1]_f \approx [X;E_2]_{g\circ f}$$

PROOF The fact that $g''_{\#}$ is surjective follows immediately from lemma 11. The fact that $g''_{\#}$ is injective follows from application of lemma 11 to the relative CW complex $(X,A) \times (I,\dot{I})$. ∎

EXERCISES

A EXACTNESS OF HOMOTOPY SETS

1 Assume that $j\colon (X',A') \subset (X,A)$ is a cofibration, where A and X' are closed subsets of X and $A' = A \cap X'$. Prove that the collapsing map

$$(C_{j'},C_{j''}) \to (C_{j'},C_{j''})/CX' = (X,A)/X' = (X/X',\ A/A')$$

is a homotopy equivalence.

2 With the same hypotheses as in exercise 1, let $g'\colon (X,A) \to C(X',A')$ be any map such that $g'(x') = x'$ for $x' \in X'$ and let $g\colon (X/X',A/A') \to S(X',A')$ be the map such that the following square is commutative, where k' and k'' are the collapsing maps:

$$\begin{array}{ccc} (X,A) & \xrightarrow{g'} & C(X',A') \\ {\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle k''} \\ (X/X',A/A') & \xrightarrow{g} & S(X',A') \end{array}$$

Prove that there is a coexact sequence

$$(X',A') \to \cdots \to S^n(X',A') \xrightarrow{S^n j} S^n(X,A) \xrightarrow{S^n k'} S^n(X/X',\ A/A') \xrightarrow{S^n g} \cdots$$

3 If (X,A) is a relative CW complex, prove that there is a coexact sequence

$$A \subset X \to X/A \to SA \subset SX \to \cdots \to S^nA \subset S^nX \to \cdots$$

B HOMOTOPY GROUPS

1 If A is a retract of X, prove that there is an isomorphism

$$\pi_n(X,x_0) \approx \pi_n(A,x_0) \oplus \pi_n(X,A,x_0) \qquad n \geq 2$$

2 If X is deformable into A relative to $x_0 \in A$, prove that there is an isomorphism

$$\pi_n(A,x_0) \approx \pi_n(X,x_0) \oplus \pi_{n+1}(X,A,x_0) \qquad n \geq 2$$

3 If $p\colon E \to B$ is a weak fibration such that the fiber $F = p^{-1}(b_0)$ is contractible in E relative to $e_0 \in F$, prove that there is an isomorphism

$$\pi_n(B,b_0) \approx \pi_n(E,e_0) \oplus \pi_{n-1}(F,e_0) \qquad n \geq 2$$

4 If $p\colon E \to B$ is a weak fibration which admits a section, prove that there is an isomorphism for $e_0 \in F = p^{-1}(b_0)$

$$\pi_n(E,e_0) \approx \pi_n(B,b_0) \oplus \pi_n(F,e_0) \qquad n \geq 2$$

5 Let $\{X_j\}$ be an indexed family of spaces with base points $x_j \in X_j$. Prove that there is an isomorphism

$$\pi_n(\times X_j,(x_j)) \approx \times \pi_n(X_j,x_j) \qquad n \geq 0$$

6 Given $X \vee Y = X \times y_0 \cup x_0 \times Y \subset X \times Y$, prove that there is an isomorphism

$$\pi_n(X \vee Y, (x_0,y_0)) \approx \pi_n(X,x_0) \oplus \pi_n(Y,y_0) \oplus \pi_{n+1}(X \times Y, X \vee Y, (x_0,y_0))$$

C BASE POINTS[1]

1 Give an example of a degenerate base point.

2 If X and Y have nondegenerate base points, prove that also $X \vee Y$, $X \times Y$, and $X \times Y/X \vee Y$ have nondegenerate base points.

3 If (X,x_0) and (Y,y_0) have the same homotopy type, prove that x_0 is a nondegenerate base point of X if and only if y_0 is a nondegenerate base point of Y.

4 Prove that any space has the same homotopy type as some space with a nondegenerate base point.

5 Let X and Y be path-connected spaces with nondegenerate base points x_0 and y_0, respectively. Prove that X and Y have the same homotopy type if and only if (X,x_0) and (Y,y_0) have the same homotopy type.

D THE WHITEHEAD PRODUCT

Let $p \geq 1$ and $q \geq 1$ and let $h\colon (I^{p+q},\dot{I}^{p+q}) \to (I^p,\dot{I}^p) \times (I^q,\dot{I}^q)$ be the homeomorphism $h(t_1, \ldots ,t_{p+q}) = ((t_1, \ldots ,t_p),(t_{p+1}, \ldots ,t_{p+q}))$. Then h determines an element $[h] \in \pi_{p+q}((I^p,\dot{I}^p) \times (I^q,\dot{I}^q), (0,0))$ and an element

$$\eta_{p,q} = \partial[h] \in \pi_{p+q-1}(I^p \times \dot{I}^q \cup \dot{I}^p \times I^q, (0,0))$$

Given maps $\alpha\colon (I^p,\dot{I}^p) \to (X,x_0)$ and $\beta\colon (I^q,\dot{I}^q) \to (X,x_0)$, define a map $\gamma\colon (I^p \times \dot{I}^q \cup \dot{I}^p \times I^q, (0,0)) \to (X,x_0)$ by

$$\gamma(z,z') = \begin{cases} \alpha(z) & z' \in \dot{I}^q,\ (z,z') \in I^p \times I^q \\ \beta(z') & z \in \dot{I}^p,\ (z,z') \in I^p \times I^q \end{cases}$$

1 Prove that $\gamma_\#(\eta_{p,q}) \in \pi_{p+q-1}(X,x_0)$ depends only on $[\alpha]$ and $[\beta]$. It is called the *Whitehead product* of $[\alpha]$ and $[\beta]$ and is denoted by $[[\alpha],[\beta]] \in \pi_{p+q-1}(X,x_0)$.

2 Prove that if $p = q = 1$, then $[[\alpha],[\beta]] = [\alpha][\beta][\alpha]^{-1}[\beta]^{-1}$.

3 If $p > 1$ and $q = 1$, prove that $[[\alpha],[\beta]] = [\alpha]h_{[\beta]}([\alpha]^{-1})$.

4 If $p + q > 2$, prove that $[[\alpha],[\beta]] = (-1)^{pq}[[\beta],[\alpha]]$.

5 If $f\colon (X,x_0) \to (Y,y_0)$, prove that $f_\#[[\alpha],[\beta]] = [f_\#[\alpha],f_\#[\beta]]$.

6 If ω is a path in X, prove that $h_{[\omega]}[[\alpha],[\beta]] = [h_{[\omega]}[\alpha],h_{[\omega]}[\beta]]$.

7 Prove that $[[\alpha],[\beta]] = 0$ if and only if there is a map $f\colon I^p \times I^q \to X$ such that

$$f(t_1, \ldots ,f_{p+q}) = \begin{cases} \alpha(t_1, \ldots ,t_p) \text{ if } t_i = 0 \text{ or } 1 \text{ for some } p+1 \leq i \leq p+q \\ \beta(t_{p+1}, \ldots ,t_{p+q}) \text{ if } t_i = 0 \text{ or } 1 \text{ for some } 1 \leq i \leq p \end{cases}$$

8 If X is an H space, prove that $[[\alpha],[\beta]] = 0$ for all $[\alpha]$ and $[\beta]$.

[1] See D. Puppe, Homotopiemengen und ihre induzierten Abbildungen. I, *Mathematische Zeitschriften*, vol. 69, pp. 299–344, 1958.

9 Prove that S^n is an H space if and only if $[[\alpha],[\beta]] = 0$ for all $[\alpha]$, $[\beta] \in \pi_n(S^n)$.

E CW COMPLEXES

1 If (X,A) is a relative CW complex, prove that X has a topology coherent with the collection $\{A\} \cup \{e \mid e \text{ a cell of } X - A\}$.

2 If (X,A) is a relative CW complex, prove that X is compactly generated if and only if A is compactly generated.

3 If (X,A) is a relative CW complex and A is paracompact, prove that X is paracompact.

4 If (X,A) is a relative CW complex and A has the same homotopy type as a CW complex, prove that X has the same homotopy type as a CW complex.

5 Prove that a CW complex is locally contractible.

6 Prove that a CW complex has the same homotopy type as a polyhedron.

F ACTION OF THE FUNDAMENTAL GROUP

1 Prove that the real projective n-space P^n is simple if and only if n is odd.

2 For $1 < n < m$ show that $P^{2n+1} \times S^{2m+1}$ and $P^{2m+1} \times S^{2n+1}$ are simple compact polyhedra having isomorphic homotopy groups in all dimensions, but are not of the same homotopy type.

3 Let $(Z,\dot{Z})$ be an $(n-1)$-connected CW pair, with $n \geq 2$, such that $\dot{Z}$ is simply connected. Let (X^*,X) be the adjunction space obtained by adjoining Z to a CW complex X by a map $f\colon (\dot{Z},z_0) \to (X,x_0)$ and let $g\colon (Z,\dot{Z},z_0) \to (X^*,X,x_0)$ be the canonical map. Prove that (X^*,X) is $(n-1)$-connected and that the map

$$\bigoplus_{[\omega] \in \pi_1(X,x_0)} [\pi_n(Z,\dot{Z},z_0)]_{[\omega]} \to \pi_n(X^*,X,x_0)$$

sending $[\alpha]_{[\omega]}$ to $h_{[\omega]}(g_\#[\alpha])$ for $[\alpha] \in \pi_n(Z,\dot{Z},z_0)$ is an isomorphism. [*Hint:* Let $\tilde{X}$ be the universal covering space of X and let $\{f_{[\omega]}\colon \dot{Z} \to \tilde{X}\}_{[\omega] \in \pi_1(X,x_0)}$ be the set of liftings of f. Show that the space $\tilde{X}^*$ obtained by attaching a copy of Z to $\tilde{X}$ for each map $f_{[\omega]}$ is the universal covering space of X^*. Then use the fact that $\pi_q(\tilde{X}^*,\tilde{X}) \approx \pi_q(X^*,X)$ and compute $\pi_n(\tilde{X}^*,\tilde{X})$ by the Hurewicz theorem.]

4 Let X be the CW complex obtained from $S^1 \vee S^2$ by attaching a 3-cell by a map representing $2[\alpha] - h_{[\omega]}[\alpha]$, where $[\alpha]$ is a generator of $\pi_2(S^2)$ and $[\omega]$ is a generator of $\pi_1(S^1)$. Prove that the inclusion map $S^1 \subset X$ induces an isomorphism of the fundamental groups and all homology groups but not of the two-dimensional homotopy groups.

G CW APPROXIMATIONS

1 If (X,A) is an arbitrary pair, prove that there is a CW pair (X',A') and a map $f\colon (X',A') \to (X,A)$ such that $f \mid X'\colon X' \to X$ and $f \mid A'\colon A' \to A$ are both weak homotopy equivalences.

2 If $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ are weak homotopy equivalences, prove that $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is also a weak homotopy equivalence.

3 If $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ are weak homotopy equivalences, show by an example that $f_1 \vee f_2\colon X_1 \vee X_2 \to Y_1 \vee Y_2$ need not be a weak homotopy equivalence.

4 Show by an example that a weak homotopy equivalence need not induce isomorphisms of the corresponding Alexander cohomology groups.

5 If X is simply connected and $H_*(X)$ is finitely generated, prove that X has the same weak homotopy type as some finite CW complex.

6 A space X is said to be *dominated* by a space Y if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq 1_X$. Prove that a space is dominated by a CW complex if and only if it has the same homotopy type as some CW complex.

H GROUPS OF HOMOTOPY CLASSES

Throughout this group of exercises it is assumed that Y is $(n-1)$-connected, where $n \geq 2$, with base point y_0, and that X is a CW complex of dimension $\leq 2n-2$.

1 Prove that any map $X \to Y$ is homotopic to a map sending X^{n-1} to y_0 and that if $f, g\colon (X,X^{n-1}) \to (Y,y_0)$ are homotopic as maps from X to Y, they are homotopic relative to X^{n-2}.

2 Prove that the diagonal map $d\colon X \to X \times X$ is homotopic to a map d' such that $d'(X) \subset (X \times X^{n-2}) \cup (X^{n-2} \times X)$. Prove that maps $d', d''\colon X \to (X \times X^{n-2}) \cup (X^{n-2} \times X)$ which are homotopic in $X \times X$ are homotopic in $(X \times X^{n-1}) \cup (X^{n-1} \times X)$. (*Hint:* Use the cellular-approximation theorem.)

Let $d'\colon X \to (X \times X^{n-2}) \cup (X^{n-2} \times X)$ be homotopic in $X \times X$ to the diagonal map. Given $f, g\colon X \to Y$, let $f', g'\colon (X,X^{n-1}) \to (Y,y_0)$ be homotopic to f and g, respectively. Then $(f' \times g') \circ d'\colon X \to Y \times Y$ maps X into $Y \vee Y$. Let $\gamma\colon Y \vee Y \to Y$ be defined by $\gamma(y,y_0) = y = \gamma(y_0,y)$.

3 Prove that $[\gamma \circ (f' \times g') \circ d']$ depends only on $[f]$ and $[g]$ and that the operation $[f] + [g] = [\gamma \circ (f' \times g') \circ d']$ is associative, commutative, and has a unit element, making $[X;Y]$ into a commutative semigroup with unit.

4 Prove that if $g\colon Y \to Y'$, where Y' is also $(n-1)$-connected (or if $h\colon X' \to X$, where X' is a CW complex of dimension $\leq 2n-2$), then $g_\#\colon [X;Y] \to [X;Y']$ is a homomorphism (or $h^\#\colon [X;Y] \to [X';Y]$ is a homomorphism).

5 The semigroup $[X;Y]$ is a group. (*Hint:* Use induction on the dimension of X, the fact that $[X^{k+1}/X^k;Y]$ is a group for any k and any Y, because X^{k+1}/X^k, being a wedge of $(k+1)$-spheres, is a suspension, and the exactness of the sequence of homomorphisms

$$[X^{k+1}/X^k;Y] \to [X^{k+1};Y] \to [X^k;Y]\ [X';Y]$$

where X' is a disjoint union of k spheres, one for each $(k+1)$-cell of X.)

In case $Y = S^n$ and dimension $X \leq 2n-2$, the group $[X;S^n]$ is called the *nth cohomotopy group of* X,[1] denoted by $\pi^n(X)$.

I MISCELLANEOUS

1 Let $\partial'\colon \pi_{n+1}(\Delta^{n+1},\dot{\Delta}^{n+1},v_0) \to \pi_n(\dot{\Delta}^{n+1},(\Delta^{n+1})^{n-1},v_0)$ if $n \geq 2$ and let

$$\partial'\colon \pi_2(\Delta^2,\dot{\Delta}^2,v_0) \to \pi_1(\dot{\Delta}^2,v_0)$$

if $n = 1$. Prove that $\partial'[\xi_{n+1}] = b_n$ for $n \geq 1$ (see page 394 for definition of b_n).

2 Let H be a homotopy functor and let $f\colon X \to Y$ be a base-point-preserving map between path-connected spaces, with nondegenerate base points. Prove that the sequence

$$H(C_f) \to H(Y) \to H(X)$$

is exact.

3 If H is a homotopy functor and (X,A) is a CW pair, prove that there is an exact sequence

$$H(A) \leftarrow H(X) \leftarrow H(X/A) \leftarrow H(SA) \leftarrow \cdots \leftarrow H(S^nA) \leftarrow \cdots$$

[1] For more details see E. Spanier, Borsuk's cohomotopy groups, *Annals of Mathematics*, vol. 50, pp. 203–245, 1949.

CHAPTER EIGHT
OBSTRUCTION THEORY

IN THIS CHAPTER WE DEVELOP OBSTRUCTION THEORY FOR THE GENERAL LIFTING problem. A sequence of obstructions is defined whose vanishing is necessary and sufficient for the existence of a lifting. The kth obstruction in the sequence is defined if and only if all the lower obstructions are defined and vanish, in which case the vanishing of the kth obstruction is a necessary condition for definition of the $(k + 1)$st obstruction.

We begin by applying the general theory of homotopy functors to study the set of homotopy classes of maps from a CW complex to a space with exactly one nonzero homotopy group and we show that a suitable cohomology functor serves to classify maps up to homotopy in this case. This result is then used to obtain a solution, in terms of cohomology, of the lifting problem for a fibration whose fiber has exactly one nonzero homotopy group.

With this in mind, we then consider the problem of factorizing an arbitrary fibration into simpler ones each of which has a fiber with exactly one nonzero homotopy group. We show that such factorizations do exist for a large class of fibrations, and that when they exist, a sequence of obstructions can be associated to the factorization. These obstructions are subsets of cohomology groups, and we apply the general machinery to some special cases where, because of dimension restrictions, the only obstructions which enter are either the first one or the first two. For the case of only one obstruction we obtain the Hopf classification theorem.

Finally, we prove the suspension theorem, which we use to compute the $(n + 1)$st homotopy group of the n-sphere. Combining this with the technique of obstruction theory, we obtain a proof of the Steenrod classification theorem.

Section 8.1 is devoted to spaces with exactly one nonzero homotopy group. We prove that a suitable cohomology functor serves both to classify maps from a CW complex to such a space and to provide a solution for the extension problem for maps involving a relative CW complex and such a space. We use this result to derive the Hopf extension and classification theorems for maps of an n-dimensional CW complex to S^n. Section 8.2 deals with fibrations whose fiber has exactly one nonzero homotopy group, and again it is shown that a suitable cohomology functor serves to provide a solution for the lifting problem and to classify liftings of a given map.

In Sec. 8.3 we prove that many fibrations can be factored as infinite composites of fibrations each of which has a fiber with exactly one nonzero homotopy group. The corresponding lifting problem is then represented as an infinite sequence of simpler lifting problems. In Sec. 8.4 we show how to define obstructions inductively for such a sequence of fibrations, and how to apply the resulting machinery.

In Sec. 8.5 we shall study the suspension map and prove the exactness of the Wang sequence of a fibration with base space a sphere. This result is used to prove the suspension theorem, which is applied to compute $\pi_{n+1}(S^n)$ for all n. We then prove the Steenrod classification theorem for maps of an $(n + 1)$-dimensional CW complex to S^n.

1 EILENBERG-MACLANE SPACES

This section is devoted to a study of spaces with exactly one nonzero homotopy group. Such spaces are classifying spaces for the cohomology functors, and because of this, there is an important relation between the cohomology of these spaces and cohomology operations. At the end of the section we shall apply the results to derive the Hopf classification and extension theorems. Then, later in the chapter, we shall study arbitrary spaces by representing them as iterated fibrations whose fibers are spaces with exactly one nonzero homotopy group. Thus, these homotopically simple spaces serve as building blocks for more complicated spaces.

Let π be a group and let n be an integer ≥ 1. A *space of type* (π,n) is a path-connected pointed space Y such that $\pi_q(Y,y_0) = 0$ for $q \neq n$ and $\pi_n(Y,y_0)$ is isomorphic to π. An *Eilenberg-MacLane space*[1] is a path-connected pointed space all of whose homotopy groups vanish, except possibly for a

[1] See S. Eilenberg and S. MacLane, On the groups $H(\pi,n)$, I, *Annals of Mathematics,* vol. 58, pp. 55–106, 1953.

single dimension. Thus a space of type (π,n) is an Eilenberg-MacLane space. Conversely, if Y is an Eilenberg-MacLane space and $\pi_q(Y,y_0) = 0$ for $q \neq n$, then Y is a space of type $(\pi_n(Y,y_0),\, n)$. Let us consider a few examples.

1 It follows from corollary 7.2.12 that S^1 is a space of type $(\mathbf{Z},1)$.

2 Let P^∞ be the CW complex which is the union of the sequence $P^1 \subset P^2 \subset \cdots$ topologized by the topology coherent with the collection $\{P^j\}_{j\geq 1}$. Then $\pi_q(P^\infty) \approx \lim_\rightarrow \{\pi_q(P^j)\}$, and it follows from application of corollary 7.2.11 to the covering $S^n \to P^n$ that P^∞ is a space of type $(\mathbf{Z}_2,1)$.

3 Let $P_\infty(\mathbf{C})$ be the CW complex which is the union of the sequence $P_1(\mathbf{C}) \subset P_2(\mathbf{C}) \subset \cdots$ topologized by the topology coherent with the collection $\{P_j(\mathbf{C})\}_{j\geq 1}$. Then $\pi_q(P_\infty(\mathbf{C})) \approx \lim_\rightarrow \{\pi_q(P_j(\mathbf{C}))\}$, and it follows from corollary 7.2.13 that $P_\infty(\mathbf{C})$ is a space of type $(\mathbf{Z},2)$.

Let π be an abelian group and Y a path-connected pointed space. An element $v \in H^n(Y,y_0;\, \pi)$ is said to be *n-characteristic* for Y if the composite

$$\pi_n(Y,y_0) \xrightarrow{\varphi} H_n(Y,y_0) \xrightarrow{h(v)} \pi$$

is an isomorphism (where φ is the Hurewicz homomorphism and h is the homomorphism defined in Sec. 5.5). If Y is $(n-1)$-connected, it follows from the absolute Hurewicz isomorphism theorem and the universal-coefficient theorem for cohomology that there is an n-characteristic element $v \in H^n(Y,y_0;\, \pi)$ if and only if $\pi \approx \pi_n(Y,y_0)$. Such an element is unique up to automorphisms of π. In particular, a space Y of type (π,n) with π abelian has n-characteristic elements $v \in H^n(Y,y_0;\, \pi)$.

4 LEMMA *Let $u \in H^n(Y,y_0;\, G)$ be a universal element for the nth cohomology functor with coefficients G, where $n \geq 1$. Then Y is a space of type (G,n) and u is n-characteristic for Y.*

PROOF By theorem 7.7.14, there are isomorphisms

$$T_u\colon \pi_q(Y,y_0) \approx H^n(S^q,p_0;\, G) \qquad q \geq 1$$

Therefore $\pi_q(Y,y_0) = 0$ if $q \neq n$, and $T_u\colon \pi_n(Y,y_0) \approx H^n(S^n,p_0;\, G)$. If $\alpha\colon (S^n,p_0) \to (Y,y_0)$, then $T_u([\alpha]) = \alpha^*(u)$, and there is a commutative diagram

$$\begin{tikzcd}
\pi_n(S^n,p_0) \arrow[r, "\varphi", "\approx"'] \arrow[dd, "\alpha_\#"'] & H_n(S^n,p_0) \arrow[dr, "h(\alpha^*(u)) = h(T_u[\alpha])"] \arrow[dd, "\alpha_*"'] & \\
 & & G \\
\pi_n(Y,y_0) \arrow[r, "\varphi", "\approx"'] & H_n(Y,y_0) \arrow[ur, "h(u)"'] &
\end{tikzcd}$$

Let $\nu\colon H^n(S^n,p_0;\, G) \approx G$ be the isomorphism defined by

$$\nu(v) = h(v)(\varphi[1_{S^n}]) \qquad v \in H^n(S^n,p_0;\, G)$$

From the commutativity of the diagram above,

$$(h(u) \circ \varphi)[\alpha] = (h(u) \circ \varphi \circ \alpha_\#)[1_{S^n}] = h(T_u[\alpha])\,(\varphi[1_{S^n}]) = (\nu \circ T_u)[\alpha]$$

It follows that $h(u) \circ \varphi$ equals the composite

$$\pi_n(Y,y_0) \xrightarrow[\approx]{T_u} H^n(S^n,p_0;\, G) \xrightarrow[\approx]{\nu} G$$

and so is an isomorphism. Therefore Y is a space of type (G,n) and u is n-characteristic for Y. ■

5 COROLLARY *Given $n \geq 1$ and a group π (abelian if $n > 1$), there exists a space of type (π,n).*

PROOF If π is abelian, it follows from lemma 4 that any classifying space for the nth cohomology functor with coefficients π is a space of type (π,n). If $n = 1$ and π is arbitrary, it is easy to see that a classifying space for the homotopy functor of example 7.7.5 which assigns to a pointed path-connected space X the set of homomorphisms $\pi_1(X,x_0) \to \pi$ is a space of type $(\pi,1)$. In either case, since any homotopy functor has a classifying space by corollary 7.7.12, the result follows. ■

6 COROLLARY *Let $\{\pi_n\}_{n\geq 1}$ be a sequence of groups which are abelian for $n \geq 2$. There is a space X, with base point x_0, such that $\pi_n(X,x_0) \approx \pi_n$ for $n \geq 1$.*

PROOF By corollary 5, for each $n \geq 1$ there is a space Y_n, with base point y_n, such that $\pi_q(Y_n,y_n) = 0$ for $q \neq n$ and $\pi_n(Y_n,y_n) \approx \pi_n$. Then the product space $\times\, Y_n$ with base point (y_n) has the desired properties. ■

The last result can be strengthened so that if π_1 acts as a group of operators on π_n for every $n \geq 2$, then the sequence is realized as the sequence of homotopy groups of a space X in such a way that the action of π_1 on π_n corresponds to the action of $\pi_1(X,x_0)$ on $\pi_n(X,x_0)$ of theorem 7.3.8.

7 LEMMA *Let $F\colon H \to H'$ be a natural transformation between homotopy functors which induces an isomorphism of their qth coefficient groups for $q < n$ and a surjection of their nth coefficient groups (where $1 \leq n \leq \infty$). For any path-connected pointed CW complex W the map*

$$F(W)\colon H(W) \to H'(W)$$

is a bijection if $\dim W \leq n - 1$ *and a surjection if* $\dim W \leq n$.

PROOF Let $u \in H(Y)$ and $u' \in H'(Y')$ be universal elements for H and H', respectively, and let $f\colon Y \to Y'$ be a map such that $H'(f)(u') = F(Y)(u)$. For any CW complex W there is a commutative square

$$\begin{array}{ccc} [W;Y] & \xrightarrow{f_*} & [W;Y'] \\ {\scriptstyle T_u}\downarrow & & \downarrow{\scriptstyle T_{u'}} \\ H(W) & \xrightarrow{F(W)} & H'(W) \end{array}$$

in which, by theorem 7.7.14, both vertical maps are bijections. Since $F(S^q)\colon H(S^q) \to H'(S^q)$ is an isomorphism for $q < n$ and a surjection for $q = n$, it follows that $f_\#\colon \pi_q(Y) \to \pi_q(Y')$ is an isomorphism for $q < n$ and a surjection for $q = n$. Since Y and Y' are path-connected pointed spaces, the map f

is an n-equivalence. The result follows from corollary 7.6.23 and the commutativity of the above square. ∎

We use this last result to obtain the following classification theorem, which is a converse of lemma 4.

8 THEOREM *Let π be an abelian group, Y a space of type (π,n), and $\iota \in H^n(Y,y_0;\ \pi)$ an n-characteristic element for Y. Let ψ: $\pi^Y \to H^n(\cdot\,;\pi)$ be the natural transformation defined by $\psi[f] = f^*\iota$ for $[f] \in [X;Y]$. Then ψ is a natural equivalence on the category of path-connected pointed CW complexes.*

PROOF By lemma 7, it suffices to verify that ψ induces an isomorphism of all coefficient groups of the two homotopy functors π^Y and $H^n(\cdot\,;\pi)$. The only nonzero coefficient groups are $\pi_n(Y,y_0)$ and $H^n(S^n,p_0;\ \pi)$, and we need only verify that

$$\psi(S^n)\colon \pi_n(Y,y_0) \to H^n(S^n,p_0;\ \pi)$$

is an isomorphism. If ν: $H^n(S^n,p_0;\ \pi) \approx \pi$ is defined by $\nu(v) = h(v)(\varphi[1_{S^n}])$ (as in the proof of lemma 4), then $\nu \circ \psi(S^n) = h(\iota) \circ \varphi$. Because ι is n-characteristic for Y, $\nu \circ \psi(S^n)$ is an isomorphism, and thus so is $\psi(S^n)$. ∎

9 THEOREM *Let Y be a space of type $(\pi,1)$ and let H be the functor which assigns to a pointed space X the set of homomorphisms from $\pi_1(X,x_0)$ to $\pi_1(Y,y_0)$. Let $\bar\psi$: $\pi^Y \to H$ be the natural transformation defined by $\bar\psi[f] = f_\#$ for $[f] \in [X;Y]$. Then $\bar\psi$ is a natural equivalence on the category of path-connected pointed CW complexes.*

PROOF By lemma 7, it suffices to verify that

$$\bar\psi(S^1)\colon \pi_1(Y,y_0) \to H(S^1,p_0)$$

is an isomorphism. Let $\bar\nu$: $H(S^1,p_0) \approx \pi_1(Y,y_0)$ be the isomorphism defined by $\bar\nu(\gamma) = \gamma([1_{S^1}])$ for γ: $\pi_1(S^1,p_0) \to \pi_1(Y,y_0)$. Then $\bar\nu$ is an inverse of $\bar\psi(S^1)$, showing that $\bar\psi(S^1)$ is an isomorphism. ∎

Note that if $\pi_1(Y,y_0)$ is abelian in theorem 9, the set of homomorphisms from $\pi_1(X,x_0)$ to $\pi_1(Y,y_0)$ is in one-to-one correspondence with the group

$$\operatorname{Hom}(\pi_1'(X,x_0),\ \pi_1(Y,y_0)) \approx \operatorname{Hom}(H_1(X,x_0),\ \pi_1(Y,y_0)) \approx H^1(X,x_0;\ \pi_1(Y,y_0))$$

and so theorems 8 and 9 agree in this case.

We now consider the free homotopy classes of maps from X to Y. Since any 0-cell x_0 of a CW complex X is a nondegenerate base point (because, by theorem 7.6.12, the inclusion map $x_0 \subset X$ is a cofibration), it follows from corollary 7.3.4 that there is an action of $\pi_1(Y,y_0)$ on the set $[X,x_0;\ Y,y_0]$. Furthermore, if Y and X are path connected and this action is trivial, then the map from base-point-preserving homotopy classes to free homotopy classes

$$[X,x_0;\ Y,y_0] \to [X;Y]$$

is a bijection. In case Y is a space of type (π,n), with $n > 1$, then $\pi_1(Y,y_0) = 0$, and so there is a bijection

$$[X,x_0;\ Y,y_0] \approx [X;Y]$$

In case Y is a space of type $(\pi,1)$, the action of $\pi_1(Y,y_0)$ on $[X,x_0;\ Y,y_0]$ corresponds under the bijection $\bar{\psi}$ of theorem 9 to the action of $\pi_1(Y,y_0)$ on $H(X,x_0)$ by conjugation. Thus, if π is abelian, there is a bijection

$$[X,x_0;\ Y,y_0] \approx [X;Y]$$

10 THEOREM *If π is an abelian group, Y is a space of type (π,n), and $\iota \in H^n(Y,y_0;\ \pi)$ is n-characteristic for Y, then for any relative CW complex (X,A) the map*

$$\psi\colon [X,A;\ Y,y_0] \to H^n(X,A;\ \pi)$$

is a bijection.

PROOF In case A is empty and X is path connected, it follows from theorem 8 and the observation above that there is a commutative square

$$\begin{array}{ccc} [X,x_0;\ Y,y_0] & \xrightarrow{\approx} & [X;Y] \\ \psi\downarrow\approx & & \downarrow\psi \\ H^n(X,x_0;\ \pi) & \xrightarrow{\approx} & H^n(X;\pi) \end{array}$$

and so $\psi\colon [X;Y] \approx H^n(X,\pi)$. In case A is empty and X is not path connected, let $\{X_\lambda\}$ be the set of path components of X. The result follows from the first case on observing that $[X;Y] \approx \times [X_\lambda;Y]$ and $H^n(X;\pi) \approx \times H^n(X_\lambda;\pi)$. In case A is not empty, let $k\colon (X,A) \to (X/A,x_0)$ be the collapsing map. Then the result follows from the already established bijection $\psi\colon [X/A;Y] \approx H^n(X/A;\pi)$ and the commutative diagram

$$\begin{array}{ccccc} [X,A;\ Y,y_0] & \xleftarrow[\approx]{k^*} & [X/A,x_0;\ Y,y_0] & \xrightarrow{\approx} & [X/A;Y] \\ \psi\downarrow & & \psi\downarrow & & \approx\downarrow\psi \\ H^n(X,A;\ \pi) & \xleftarrow[\approx]{k^*} & H^n(X/A,x_0;\ \pi) & \xrightarrow{\approx} & H^n(X/A;\pi) \end{array} \quad \blacksquare$$

11 THEOREM *Let Y be a space of type $(\pi,1)$. For any path-connected CW complex X the set of free homotopy classes of maps from X to Y is in one-to-one correspondence with the set of conjugacy classes of homomorphisms $\pi_1(X,x_0) \to \pi_1(Y,y_0)$ under the map $[f] \to f_\#$.*

PROOF This follows from theorem 9 and the remark above covering the action of $\pi_1(Y,y_0)$ on $[X,x_0;\ Y,y_0]$. ■

12 THEOREM *Let Y be a space of type (π,n), with $n \geq 1$ and π abelian, and let $\iota \in H^n(Y,y_0;\ \pi)$ be n-characteristic for Y. If (X,A) is a relative CW complex, a map $f\colon A \to Y$ can be extended over X if and only if* $\delta f^*(\iota) = 0$ in $H^{n+1}(X,A;\pi)$

PROOF Assume $f = g \circ i$, where $i\colon A \subset X$ and $g\colon X \to Y$. Then $\delta f^*(\iota) = \delta i^* g^*(\iota) = 0$, because $\delta i^* = 0$. Hence, if f can be extended over X, then $\delta f^*(\iota) = 0$.

Conversely, assume $\delta f^*(\iota) = 0$. To extend f over X we need only extend f over each path component of X, and therefore there is no loss of generality in assuming X to be path connected (and A to be nonempty). Let Y' be the space obtained from the disjoint union $X \cup Y$ by identifying $a \in A$ with $f(a) \in Y$ for all $a \in A$. Then Y is imbedded in Y', the pair (Y',Y) is a relative CW complex, and there is a cellular map j: $(X,A) \to (Y',Y)$ which induces an isomorphism $j^*: H^*(Y',Y) \approx H^*(X,A)$ such that there is a commutative square

$$\begin{array}{ccc} H^n(Y,y_0) & \xrightarrow{\delta} & H^{n+1}(Y',Y) \\ {\scriptstyle f^*}\downarrow & & \approx\downarrow{\scriptstyle j^*} \\ H^n(A) & \xrightarrow{\delta} & H^{n+1}(X,A) \end{array}$$

Since $\delta f^*(\iota) = 0$, it follows that $\delta(\iota) = 0$, and there is $v \in H^n(Y',y_0;\ \pi)$ such that $v \mid (Y,y_0) = \iota$. Since X and Y are path connected and A is nonempty, Y' is path connected.

Let $\bar{Y} = Y' \vee I$ (that is, $y_0 \in Y'$ is identified with $0 \in I$) and let $\bar{y}_0 = 1 \in \bar{Y}$. Then $\bar{Y}$ is a path-connected space with nondegenerate base point $\bar{y}_0$. Let r: $(\bar{Y},I) \to (Y',y_0)$ be the retraction which collapses I to y_0 and let $\bar{v} = r^*(v) \mid (\bar{Y},\bar{y}_0) \in H^n(\bar{Y},\bar{y}_0;\ \pi)$. By theorem 7.7.11, there is an imbedding of $\bar{Y}$ in a space Y'' which is a classifying space for the nth cohomology functor with coefficients π and which has a universal element $\bar{u} \in H^n(Y'',\bar{y}_0;\ \pi)$ such that $\bar{u}|(\bar{Y},\bar{y}_0) = \bar{v}$. Then Y'' is a space of type (π,n), and there is a unique n-characteristic element $u \in H^n(Y'',y_0;\ \pi)$ such that $u \mid Y'' = \bar{u} \mid Y''$. Then $u \mid (Y,y_0) = \iota$, and it follows from theorem 8 and the commutativity of the diagram

$$\begin{array}{ccc} [S^q,p_0;\ Y,y_0] & \to & [S^q,p_0;\ Y'',y_0] \\ {\scriptstyle \psi_\iota}\searrow\approx & & \approx\swarrow{\scriptstyle \psi_u} \\ & H^n(S^q,p_0;\ \pi) & \end{array}$$

that $Y \subset Y''$ is a weak homotopy equivalence. Since the composite $X \xrightarrow{j|X} Y' \subset Y''$ is an extension of the composite $A \xrightarrow{f} Y \subset Y''$, it follows from theorem 7.6.22 that f can be extended to a map $X \to Y$. ■

We now show that cohomology operations are closely related to the cohomology of Eilenberg-MacLane spaces. Let $\Theta(n,q;\ \pi,G)$ be the group of all cohomology operations of type $(n,q;\ \pi,G)$. Thus π and G are abelian groups and an element $\theta \in \Theta(n,q;\ \pi,G)$ is a natural transformation from the singular cohomology functor $H^n(\cdot\,;\pi)$ to the singular cohomology functor $H^q(\cdot\,;G)$.

13 THEOREM *Let π be an abelian group and let Y be a space of type (π,n), with an n-characteristic element $\iota \in H^n(Y,y_0;\ \pi)$. There is an isomorphism*

$$\gamma\colon \Theta(n,q;\ \pi,G) \approx H^q(Y,y_0;\ G)$$

defined by $\gamma(\theta) = \theta(\iota)$ for $\theta \in \Theta(n,q;\ \pi,G)$.

PROOF Since, by theorem 7.8.1, every pair has a relative CW approximation, a cohomology operation corresponds bijectively to a cohomology operation on the category of relative CW complexes. To define an inverse to γ, given $u \in H^q(Y,y_0;\, G)$, let θ_u be the cohomology operation of type $(n,q;\, \pi,G)$ defined for a relative CW complex (X,A) by

$$\theta_u(v) = f_v^*(u) \qquad v \in H^n(X,A;\, \pi)$$

where $f_v\colon (X,A) \to (Y,y_0)$ is a map such that $f_v^*(\iota) = v$ (f_v exists and is unique up to homotopy, by theorem 10). Then

$$\gamma(\theta_u) = \theta_u(\iota) = 1_Y^*(u) = u$$

showing that the map $u \to \theta_u$ is a right inverse of γ. To show that it is also a left inverse of γ, let (X,A) be a relative CW complex and let $v \in H^n(X,A;\, \pi)$. We must show that $\theta_{\gamma(\theta)}(v) = \theta(v)$. Let $f_v\colon (X,A) \to (Y,y_0)$ be such that $f_v^*(\iota) = v$. Then we have

$$\theta(v) = \theta(f_v^*(\iota)) = f_v^*(\theta(\iota)) = f_v^*(\gamma(\theta)) = \theta_{\gamma(\theta)}(v) \quad \blacksquare$$

We present one application of this result.

14 COROLLARY *Let θ be a cohomology operation of type $(n,q;\, \pi,G)$. For any relative CW complex (X,A) the map*

$$\theta\colon H^n((X,A) \times (I,\dot I);\, \pi) \to H^q((X,A) \times (I,\dot I);\, G)$$

is a homomorphism.

PROOF The collapsing map

$$k\colon (X \times I,\, A \times I \cup X \times \dot I) \to X \times I/(A \times I \cup X \times \dot I)$$

induces isomorphisms in cohomology. Furthermore, $X \times I/(A \times I \cup X \times \dot I)$ is homeomorphic to $S(X/A)$ (where X/A is understood to be the disjoint union of X and a base point x_0 in case A is empty). Thus it suffices to show that if X' is any pointed CW complex, then the map

$$\theta\colon H^n(SX',x_0';\, \pi) \to H^q(SX',x_0';\, G)$$

is a homomorphism.

Let Y be a CW complex of type (π,n), with n-characteristic element ι, and let Y' be a space of type (G,q), with q-characteristic element ι'. Let $f\colon Y \to Y'$ be a map such that $f^*\iota' = \theta(\iota)$. There is then a commutative diagram

$$\begin{array}{ccc} [SX',x_0';\, Y,y_0] & \xrightarrow{f_\#} & [SX',x_0';\, Y',y_0'] \\ {\scriptstyle\psi}\downarrow\approx & & \approx\downarrow{\scriptstyle\psi} \\ H^n(SX',x_0';\, \pi) & \xrightarrow{\theta} & H^q(SX',x_0';\, G) \end{array}$$

It is trivial that $f_\#$ is a homomorphism when the top two sets are given group structures by the H cogroup structure of SX'. By lemma 7.7.6, it follows that both vertical maps are homomorphisms. Hence the bottom map θ is a homomorphism. ■

Let $\bar{I} \in H^1(I,\dot{I};\ Z)$ be a generator and define an isomorphism

$$\tau\colon H^r(X,A;\ G') \approx H^{r+1}((X,A) \times (I,\dot{I});\ G')$$

by $\tau(u) = u \times \bar{I}$. Given a cohomology operation θ of type $(n,q;\ \pi,G)$, its *suspension* $S\theta$ is the cohomology operation of type $(n-1,\ q-1;\ \pi,G)$ defined by $(S\theta)(u) = \tau^{-1}\theta\tau(u)$ for $u \in H^{n-1}(X,A;\ \pi)$. Then corollary 14 implies that the suspension of any cohomology operation is an additive cohomology operation.

We now extend theorems 10 and 12 to other spaces Y by restricting the dimension of the relative CW complex (X,A). Let Y be an n-simple $(n-1)$-connected pointed space for some $n \geq 1$ [if $n = 1$ then $\pi_1(Y,y_0)$ is abelian]. If $\iota \in H^n(Y,y_0;\ \pi)$ is an n-characteristic element for Y, an argument similar to that in theorem 12 shows that Y can be imbedded in a space Y' of type (π,n) having an n-characteristic element $u \in H^n(Y',y_0;\ \pi)$ such that $u \mid Y = \iota$. It follows that the inclusion map $Y \subset Y'$ is an $(n+1)$-equivalence. Then theorems 7.6.22 and 10 yield the following generalization of theorem 10.

15 THEOREM *Let $\iota \in H^n(Y,y_0;\ \pi)$ be n-characteristic for an n-simple $(n-1)$-connected pointed space Y and let (X,A) be a relative CW complex. The map*

$$\psi_\iota\colon [X,A;\ Y,y_0] \to H^n(X,A;\ \pi)$$

defined by $\psi_\iota[f] = f^*(\iota)$ is a bijection if $\dim(X-A) \leq n$ *and a surjection if* $\dim(X-A) \leq n+1$. ■

For the special case $Y = S^n$ let $s^* \in H^n(S^n,p_0;\ \mathbf{Z})$ be a generator. Then s^* is an n-characteristic element of S^n, and we obtain the following *Hopf classification theorem.*[1]

16 COROLLARY *Let (X,A) be a relative CW complex, with* $\dim(X-A) \leq n$, *where $n \geq 1$. If $s^* \in H^n(S^n,p_0;\ \mathbf{Z})$ is a generator, there is a bijection*

$$\psi_{s^*}\colon [X,A;\ S^n,p_0] \approx H^n(X,A;\ \mathbf{Z})$$

defined by $\psi_{s^*}([f]) = f^*(s^*)$. ■

Similarly, we obtain the following generalization of theorem 12.

17 THEOREM *Let $\iota \in H^n(Y,y_0;\ \pi)$ be n-characteristic for an n-simple $(n-1)$-connected pointed space Y and let (X,A) be a relative CW complex, with* $\dim(X-A) \leq n+1$. *A map $f\colon A \to Y$ can be extended over X if and only if $\delta f^*(\iota) = 0$ in $H^{n+1}(X,A;\ \pi)$.* ■

This specializes to the following *Hopf extension theorem.*

[1] See H. Hopf, Die Klassen der Abbildungen der n-dimensionalen Polyeder auf die n-dimensionale Sphäre, *Commentarii Mathematici Helvetici*, vol. 5, pp. 39–54, 1933, and H. Whitney, The maps of an n-complex into an n-sphere, *Duke Mathematical Journal*, vol. 3, pp. 51–55, 1937.

18 COROLLARY *Let (X,A) be a relative CW complex, with* $\dim(X - A) \leq n + 1$, *and let* $s^* \in H^n(S^n,p_0; \mathbf{Z})$ *be a generator. A map* $f: A \to S^n$ *can be extended over X if and only if* $\delta f^*(s^*) = 0$ *in* $H^{n+1}(X,A; \mathbf{Z})$. ∎

2 PRINCIPAL FIBRATIONS

This section is concerned with fibrations whose fiber is an Eilenberg-MacLane space. We shall develop an obstruction theory for the lifting problem of maps of relative CW complexes to such fibrations. In the next section we shall show that many maps can be factored up to weak homotopy type as infinite composites of such fibrations. In this way the obstruction theory for these special fibrations leads to an obstruction theory for arbitrary maps.

For any pointed space B' there is the path fibration $PB' \xrightarrow{p'} B'$, where PB' is the space of paths in B' beginning at the base point b_0'. Under the exponential correspondence there is a one-to-one correspondence between homotopies $H: X \times I \to B'$ such that $H(x,0) = b_0'$ and maps $H': X \to PB'$, the correspondence defined by $H'(x)(t) = H(x,t)$. This easily implies the following result (which is dual to lemma 7.1.1).

1 LEMMA *A map $X \to B'$ is null homotopic if and only if it can be lifted to the path fibration $PB' \to B'$.* ∎

If $\theta: B \to B'$ is a base-point-preserving map, there is a fibration $p_\theta: E_\theta \to B$ induced from the path fibration $PB' \to B'$. This induced fibration is called the *principal fibration induced by* θ and has fiber $p_\theta^{-1}(b_0) = b_0 \times \Omega B'$. A straightforward verification shows that there is a covariant functor from the category of base-point-preserving maps between pointed spaces to the subcategory of fibrations which assigns to θ the principal fibration induced by θ.

Let (X,A) be a pair and let $i: A \subset X$ be the inclusion map. Let $p_\theta: E_\theta \to B$ be the principal fibration induced by $\theta: B \to B'$. Recall that a map pair $f: i \to p_\theta$ (defined in Sec. 7.8) is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f''} & E_\theta \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_\theta} \\ X & \xrightarrow{f'} & B \end{array}$$

The set of homotopy classes $[i;p_\theta]$ of map pairs from i to p_θ is the object function of a functor of two variables contravariant in pairs (X,A) and covariant in base-point-preserving maps θ. We are interested in studying in more detail the relative-lifting problem (that is, the map $\rho: [X;E_\theta] \to [i;p_\theta]$) for this situation. Because p_θ is an induced fibration, the relative-lifting problem is equivalent to an extension problem, as shown below.

Let $p_\theta: E_\theta \to B$ be induced by $\theta: B \to B'$. For any space W a map $f: W \to E_\theta$ consists of a pair $f_1: W \to B$ and $f_2: W \to PB'$ such that $p' \circ f_2 = \theta \circ f_1$. By the exponential correspondence, f_2 corresponds to a homotopy F: $W \times I \to B'$ from the constant map to $\theta \circ f_1$. Thus, given a map $f_1: W \to B$, there is a one-to-one correspondence between liftings $f: W \to E_\theta$

of f_1 and homotopies $F\colon W \times I \to B'$ from the constant map to $\theta \circ f_1$.

Let (X,A) be a pair with inclusion map $i\colon A \subset X$ and let $f\colon i \to p_\theta$ be a map pair consisting of maps $f''\colon A \to E_\theta$ and $f'\colon X \to B$ such that $p_\theta \circ f'' = f' \circ i$. We define a map

$$\theta(f)\colon (A \times I \cup X \times \dot{I},\, X \times 0) \to (B',b_0')$$

by the conditions $\theta(f)(x,0) = b_0'$, $\theta(f)(x,1) = \theta f'(x)$, for $x \in X$, and $\theta(f) \mid A \times I$ is the homotopy from the constant map $A \to b_0'$ to the map $\theta \circ f' \circ i$ corresponding to the lifting f'' of $f' \circ i$. There is then a one-to-one correspondence between liftings of f and extensions of $\theta(f)$ over $X \times I$.

We now specialize to the case where B' is a space of type (π,n), with $n \geq 1$ and π abelian, and we let $\iota \in H^n(B',b_0';\, \pi)$ be n-characteristic for B'. In this case, if $\theta\colon B \to B'$ is a base-point-preserving map, the induced fibration $p_\theta\colon E_\theta \to B$ is called a *principal fibration of type* (π,n). If (X,A) is a relative CW complex, then $(X,A) \times (I,\dot{I})$ is also a relative CW complex, and given a map $g\colon A \times I \cup X \times \dot{I} \to B'$, it follows from theorem 8.1.12 that g can be extended over $X \times I$ if and only if $\delta g^*(\iota) = 0$ in $H^{n+1}((X,A) \times (I,\dot{I});\, \pi)$. In particular, given a map pair $f\colon i \to p_\theta$, there is a lifting of f if and only if $\delta\theta(f)^*(\iota) = 0$. The *obstruction to lifting* f, denoted by $c(f) \in H^n(X,A;\, \pi)$, is defined by

$$\delta\theta(f)^*(\iota) = (-1)^n \tau(c(f))$$

where $\tau\colon H^n(X,A;\, \pi) \approx H^{n+1}((X,A) \times (I,\dot{I});\, \pi)$ is the map $\tau(u) = u \times \bar{I}$, defined in Sec. 8.1 [$\bar{I} \in H^1(I,\dot{I};\, \mathbf{Z})$ is the generator such that if $\bar{0} \in H^0(\{0\};\, \mathbf{Z})$ and $\bar{1} \in H^0(\{1\};\, \mathbf{Z})$ are the respective unit integral cohomology classes, then, identifying $H^0(\dot{I};\mathbf{Z}) \approx H^0(\{0\};\mathbf{Z}) \oplus H^0(\{1\};\mathbf{Z})$, we have $\delta\bar{1} = \bar{I} = -\delta\bar{0}$].

2 EXAMPLE In case A is empty, a map pair $f\colon i \to p_\theta$ is just a map $f'\colon X \to B$. In this case $\theta(f)\colon X \times \dot{I} \to B'$ is such that $\theta(f)(x,0) = b_0'$ and $\theta(f)(x,1) = \theta f'(x)$. Then $\theta(f)^*(\iota) = f'^*\theta^*(\iota) \times \bar{1}$, and so, by statement 5.6.6,

$$\delta\theta(f)^*(\iota) = (-1)^n f'^*\theta^*(\iota) \times \bar{I} = (-1)^n \tau f'^*\theta^*(\iota)$$

Therefore, in this case $c(f) = f'^*\theta^*(\iota)$.

It is clear from the definition that the obstruction to lifting f is functorial in i and θ and that it vanishes if and only if there is a lifting of f. We obtain a similar cohomological criterion for the existence of a homotopy relative to f of two liftings of f.

Let $f\colon i \to p_\theta$ be a map pair, where (X,A) is a relative CW complex, with $i\colon A \subset X$, and p_θ is a principal fibration of type (π,n). Given two liftings $\bar{f}_0, \bar{f}_1\colon X \to E_\theta$ of f, let $g\colon i' \to p_\theta$ be the map pair consisting of the commutative square

$$\begin{array}{ccc} A \times I \cup X \times \dot{I} & \xrightarrow{g''} & E_\theta \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle p_\theta} \\ X \times I & \xrightarrow{g'} & B \end{array}$$

where g' is the composite $X \times I \to X \xrightarrow{f'} B$ and g'' is the map such that $g''(x,0) = \bar{f}_0(x)$ and $g''(x,1) = \bar{f}_1(x)$ for $x \in X$ and $g''(a,t) = f''(a)$ for $a \in A$ and $t \in I$. Then $\bar{f}_0$ and $\bar{f}_1$ are homotopic relative to f if and only if g can be lifted. The obstruction to lifting g is an element $c(g) \in H^n((X,A) \times (I,\dot{I}); \pi)$, and we define the *difference between* $\bar{f}_0$ *and* $\bar{f}_1$, denoted by $d(\bar{f}_0,\bar{f}_1) \in H^{n-1}(X,A; \pi)$, by

$$c(g) = (-1)^n \tau(d(\bar{f}_0,\bar{f}_1))$$

[so $\delta\theta(g)^*(\iota) = \tau^2(d(\bar{f}_0,\bar{f}_1))$]. Then $\bar{f}_0$ and $\bar{f}_1$ are homotopic relative to f if and only if $d(\bar{f}_0,\bar{f}_1) = 0$. The difference $d(\bar{f}_0,\bar{f}_1)$ is functorial and has the following fundamental properties.

3 LEMMA *Given a map pair $f\colon i \to p_\theta$ and liftings $\bar{f}_0, \bar{f}_1, \bar{f}_2\colon X \to E_\theta$, then*

$$d(\bar{f}_0,\bar{f}_2) = d(\bar{f}_0,\bar{f}_1) + d(\bar{f}_1,\bar{f}_2)$$

PROOF Let $I_1 = [0,\frac{1}{2}]$, $\dot{I}_1 = \{0,\frac{1}{2}\}$, $I_2 = [\frac{1}{2},1]$, and $\dot{I}_2 = \{\frac{1}{2},1\}$ and define a map pair $G\colon \bar{i} \to p_\theta$ consisting of the commutative square

$$\begin{array}{ccc} A \times I \cup X \times (\dot{I}_1 \cup \dot{I}_2) & \xrightarrow{G''} & E_\theta \\ \bar{i}\downarrow & & \downarrow p_\theta \\ X \times I & \xrightarrow{G'} & B \end{array}$$

where $G'(x,t) = f'(x)$, $G''(a,t) = f''(a)$, $G''(x,0) = \bar{f}_0(x)$, $G''(x,\frac{1}{2}) = \bar{f}_1(x)$, and $G''(x,1) = \bar{f}_2(x)$. Then $c(G) \in H^n((X,A) \times (I, \dot{I}_1 \cup \dot{I}_2); \pi)$, and by the naturality of $c(G)$ and the definition of d, we see that

$$\begin{aligned} c(G) \mid (X,A) \times (I,\dot{I}) &= (-1)^n \tau(d(\bar{f}_0,\bar{f}_2)) \\ c(G) \mid (X,A) \times (I_1,\dot{I}_1) &= (-1)^n \tau_1(d(\bar{f}_0,\bar{f}_1)) \\ c(G) \mid (X,A) \times (I_2,\dot{I}_2) &= (-1)^n \tau_2(d(\bar{f}_1,\bar{f}_2)) \end{aligned}$$

where $$\tau_1\colon H^{n-1}(X,A) \approx H^n((X,A) \times (I_1,\dot{I}_1))$$

and $$\tau_2\colon H^{n-1}(X,A) \approx H^n((X,A) \times (I_2,\dot{I}_2))$$

are defined analogously to τ. From these properties, an argument similar to that used in proving that the Hurewicz homomorphism is a homomorphism (cf. theorem 7.4.3) shows that

$$\tau(d(\bar{f}_0,\bar{f}_2)) = \tau(d(\bar{f}_0,\bar{f}_1)) + \tau(d(\bar{f}_1,\bar{f}_2))$$

Since τ is an isomorphism, this is the result. ∎

4 THEOREM *Given a map pair $f\colon i \to p_\theta$, a lifting $\bar{f}_0\colon X \to E_\theta$ of f, and an element $v \in H^{n-1}(X,A; \pi)$, there is a lifting $\bar{f}_1\colon X \to E_\theta$ of f such that $d(\bar{f}_0,\bar{f}_1) = v$.*

PROOF The map $\theta(f)\colon A \times I \cup X \times \dot{I} \to B'$ used in defining $c(f)$ admits an extension $h_0\colon X \times I \to B'$ which corresponds to the lifting $\bar{f}_0\colon X \to E_\theta$. We seek another extension of $\theta(f)$ which will correspond to the desired lifting $\bar{f}_1$ of f. Let $F\colon (A \times I \times I \cup X \times (0 \times I \cup I \times \dot{I}), X \times I \times 0) \to (B',b_0')$ be the map defined by $F(a,t,t') = \theta(f)(a,t')$ for $a \in A$ and t, $t' \in I$, and $F(x,0,t) = h_0(x,t)$, $F(x,t,0) = b_0'$, and $F(x,t,1) = h_0(x,1)$ for $x \in X$ and $t \in I$.

Because $X \times I \times 0$ is a strong deformation retract of the space $A \times I \times I \cup X \times (0 \times I \cup I \times \dot{I})$, there is a homotopy relative to $X \times I \times 0$ from F to the constant map F' from $A \times I \times I \cup X \times (0 \times I \cup I \times \dot{I})$ to b_0'.

Let $G\colon (X \times 1 \times I, A \times 1 \times I \cup X \times 1 \times \dot{I}) \to (B',b_0')$ be a map such that $G^*(\iota) = (-1)^{n-1}v \times \bar{1} \times \bar{I} \in H^n((X,A) \times \{1\} \times (I,\dot{I}); \pi)$ [such a map exists, by theorem 8.1.10, because $(X,A) \times \{1\} \times (I,\dot{I})$ is a relative CW complex]. There is a well-defined map

$$H'\colon (A \times I^2 \cup X \times \dot{I}^2, A \times I \times I \cup X \times (0 \times I \cup I \times \dot{I})) \to (B',b_0')$$

such that $H' \mid X \times 1 \times I = G$. Then

$$H' \mid A \times I \times I \cup X \times (0 \times I \cup I \times \dot{I}) = F'$$

and because $(X,A) \times (I \times I, 0 \times I \cup I \times \dot{I})$ is a relative CW complex, the homotopy $F' \simeq F \operatorname{rel} X \times I \times 0$ extends to a homotopy $H' \simeq H \operatorname{rel} X \times I \times 0$, where

$$H\colon (A \times I \times I \cup X \times \dot{I} \times I \cup X \times I \times \dot{I}, X \times I \times 0) \to (B',b_0')$$

is an extension of F. Let $h_1\colon X \times I \to B'$ be defined by $h_1(x,t) = H(x,1,t)$. Since H is an extension of F, h_1 is an extension of $\theta(f)$, and hence h_1 corresponds to a lifting $\bar{f}_1$ of f.

We now show that $\bar{f}_1$ has the desired properties. The definition of the map pair $g\colon i' \to p_\theta$ used to define $d(\bar{f}_0,\bar{f}_1)$ is such that $\theta(g) = H$. Therefore

$$\tau^2(d(\bar{f}_0,\bar{f}_1)) = \delta H^*(\iota) = \delta H'^*(\iota)$$

H' is a map from $(A \times I^2 \cup X \times \dot{I}^2, A \times I^2 \cup X \times (0 \times I \cup I \times \dot{I}))$ to (B',b_0') whose restriction to $X \times 1 \times I$ is G. From the commutativity of the diagram [where the map μ is given by $\mu(w \times \bar{1} \times \bar{I}) = w \times \bar{I}$ for $w \in H^*(X,A)$]

$$\begin{tikzcd}
& H^n(A \times I^2 \cup X \times \dot{I}^2, A \times I^2 \cup X \times (0 \times I \cup I \times \dot{I})) \arrow[dl, "\approx"'] \arrow[dr, "\approx"] & \\
H^n(A \times I^2 \cup X \times \dot{I}^2, X \times I \times 0) \arrow[d, "\delta"'] & & H^n(X \times 1 \times I, A \times 1 \times I \cup X \times 1 \times \dot{I}) \arrow[d, "\mu"', "\approx"] \\
H^{n+1}((X,A) \times (I^2,\dot{I}^2)) & & H^n((X,A) \times (I,\dot{I})) \arrow[ll, "(-1)^{n-1}\tau"']
\end{tikzcd}$$

it follows that

$$\delta H'^*(\iota) = (-1)^{n-1}\tau\mu G^*(\iota) = \tau(v \times \bar{I}) = \tau^2(v)$$

Since τ^2 is an isomorphism, $d(\bar{f}_0,\bar{f}_1) = v$. ■

5 THEOREM *Let (X,A) be a relative CW complex and let (X',A) be a subcomplex, with inclusion maps $i\colon A \subset X$, $i'\colon A \subset X'$, and $i''\colon X' \subset X$. Given a map pair $f\colon i \to p_\theta$ (consisting of $f''\colon A \to E_\theta$ and $f'\colon X \to B$) and two liftings $\bar{g}_0, \bar{g}_1\colon X' \to E_\theta$ of $f \mid i'\colon i' \to p_\theta$, let $g_0, g_1\colon i'' \to p_\theta$ be the map pairs consisting, respectively, of the commutative squares*

$$\begin{array}{ccc} X' & \xrightarrow{\bar{g}_0} & E_\theta \\ {\scriptstyle i''}\downarrow & & \downarrow{\scriptstyle p_\theta} \\ X & \xrightarrow{f'} & B \end{array} \qquad \begin{array}{ccc} X' & \xrightarrow{\bar{g}_1} & E_\theta \\ {\scriptstyle i''}\downarrow & & \downarrow{\scriptstyle p_\theta} \\ X & \xrightarrow{f'} & B \end{array}$$

Then

$$\delta d(\bar{g}_0,\bar{g}_1) = c(g_0) - c(g_1)$$

where $\delta: H^{n-1}(X',A; \pi) \to H^n(X,X'; \pi)$.

PROOF Let $h: \bar{i} \to p_\theta$ be the map pair defined by the commutative square

$$\begin{array}{ccc} A \times I \cup X' \times \dot{I} & \xrightarrow{h''} & E_\theta \\ {\scriptstyle \bar{i}}\downarrow & & \downarrow{\scriptstyle p_\theta} \\ X' \times I \cup X \times \dot{I} & \xrightarrow{h'} & B \end{array}$$

where $h''(a,t) = f''(a)$ for $a \in A$ and $t \in I$, $h''(x',0) = \bar{g}_0(x')$ and $h''(x',1) = \bar{g}_1(x')$ for $x' \in X'$, and $h'(x,t) = f'(x)$ for $(x,t) \in X' \times I \cup X \times \dot{I}$. Then $c(h) \in H^n(X' \times I \cup X \times \dot{I}, A \times I \cup X' \times \dot{I}; \pi)$. There is an isomorphism

$$H^n(X' \times I \cup X \times \dot{I}, A \times I \cup X' \times \dot{I}; \pi) \approx H^n((X',A) \times (I,\dot{I}); \pi) \oplus H^n((X,X') \times \dot{I}; \pi)$$

induced by restriction. By the naturality of the obstruction, $c(h)$ corresponds to $(-1)^n\tau d(\bar{g}_0,\bar{g}_1) = (-1)^n d(\bar{g}_0,\bar{g}_1) \times \bar{I}$ in the first summand and to $c(g_0) \times \bar{0} + c(g_1) \times 1$ in the second summand.

Let $\bar{h}: \bar{\bar{i}} \to p_\theta$ be the map pair defined by the commutative square

$$\begin{array}{ccc} A \times I \cup X' \times \dot{I} & \xrightarrow{h''} & E_\theta \\ {\scriptstyle \bar{\bar{i}}}\downarrow & & \downarrow{\scriptstyle p_\theta} \\ X \times I & \xrightarrow{\bar{h}'} & B \end{array}$$

where $\bar{h}'(x,t) = f'(x)$ for $x \in X$ and $t \in I$. Then

$$c(\bar{h}) \in H^n(X \times I, A \times I \cup X' \times \dot{I}; \pi)$$

and by the naturality of the obstruction again,

$$c(\bar{h}) \mid (X' \times I \cup X \times \dot{I}, A \times I \cup X' \times \dot{I}) = c(h)$$

From the exactness of the sequence

$$H^n(X \times I, A \times I \cup X' \times \dot{I}) \to H^n(X' \times I \cup X \times \dot{I}, A \times I \cup X' \times \dot{I}) \xrightarrow{\delta} H^{n+1}(X \times I, X' \times I \cup X \times \dot{I})$$

it follows that $\delta c(h) = 0$. Therefore, in $H^{n+1}((X,A) \times (I,\dot{I}); \pi)$ we have (using theorem 5.6.6)

$$\begin{aligned} 0 &= \delta[(-1)^n d(\bar{g}_0,\bar{g}_1) \times \bar{I} + c(g_0) \times \bar{0} + c(g_1) \times \bar{1}] \\ &= (-1)^n \delta d(\bar{g}_0,\bar{g}_1) \times \bar{I} - (-1)^n c(g_0) \times \bar{I} + (-1)^n c(g_1) \times \bar{I} \end{aligned}$$

Therefore $\tau(\delta d(\bar{g}_0,\bar{g}_1) - c(g_0) + c(g_1)) = 0$, and since τ is an isomorphism, the result follows. ∎

We compute the obstruction $c(f)$ explicitly for the case of a fibration $p'\colon \Omega B' \to b_0'$, where B' is a space of type (π,n), with $n > 1$. Then $\Omega B'$ is a space of type $(\pi, n-1)$, and if $\iota' \in H^{n-1}(\Omega B',\omega_0'; \pi)$ is $(n-1)$-characteristic for $\Omega B'$ and $\iota \in H^n(B',b_0'; \pi)$ is n-characteristic for B', then $\delta\iota'$ and $p^*\iota$ [where $\delta\colon H^{n-1}(\Omega B',\omega_0') \approx H^n(PB',\Omega B')$ and $p\colon (PB',\Omega B') \to (B',b_0')$] are both elements of $H^n(PB',\Omega B'; \pi)$. The characteristic elements ι and ι' are said to be *related* if $\delta\iota' = p^*\iota$. Given one of ι or ι', it is always possible to choose the other one (uniquely) so that the two are related.

6 THEOREM *Let $\iota \in H^n(B',b_0'; \pi)$ and $\iota' \in H^{n-1}(\Omega B',\omega_0'; \pi)$ be related characteristic elements. Let (X,A) be a relative CW complex, with inclusion map $i\colon A \subset X$. Given a map pair $f\colon i \to p'$, where $p'\colon \Omega B' \to b_0'$, then $c(f) = -\delta f''^*(\iota')$, where $f''\colon A \to \Omega B'$ is part of f.*

PROOF Let $\bar f\colon (A \times I, A \times \dot I) \to (PB',\Omega B')$ be the map defined by $\bar f(a,t)(t') = f''(a)(tt')$. Then

$$\theta(f)\colon (A \times I \cup X \times \dot I, X \times 0) \to (B',b_0')$$

is the map such that $\theta(f) \mid A \times I = p \circ \bar f$ and $\theta(f)(X \times \dot I) = b_0'$. Let $\bar{\bar f}\colon (A \times I \cup X \times \dot I, X \times \dot I) \to (B',b_0')$ be the map defined by $\theta(f)$ and let $\bar f'\colon (A \times \dot I, A \times 0) \to (\Omega B',\omega_0)$ be the map defined by $\bar f$. There is then a commutative diagram [in which j and j' are appropriate inclusion maps and $h_1\colon A \to (X \times \dot I, A \times 0)$ is defined by $h_1(a) = (a,1)$]

$$\begin{tikzcd}
 & H^n(A \times I \cup X \times \dot I, X \times 0) \arrow[dr, "\delta"] & \\
H^n(B',b_0') \arrow[ur, "\theta(f)^*"] \arrow[r, "\bar{\bar f}^*"] \arrow[d, "p^*"'] & H^n(A \times I \cup X \times \dot I, X \times \dot I) \arrow[u, "j^*"'] \arrow[r, "\delta"] \arrow[d, "j'^*"'] & H^{n+1}((X,A) \times (I,\dot I)) \\
H^n(PB',\Omega B') \arrow[r, "\bar f^*"] & H^n(A \times I, A \times \dot I) & H^n(X,A) \arrow[u, "\approx"', "(-1)^{n-1}\tau"] \\
H^{n-1}(\Omega B',\omega_0') \arrow[u, "\delta"] \arrow[r, "\bar f'^*"] & H^{n-1}(A \times \dot I, A \times 0) \arrow[u, "\delta"] \arrow[r, "h_1^*", "\approx"'] & H^{n-1}(A) \arrow[u, "\delta"'] \arrow[ul, "(-1)^{n-1}\tau", "\approx"']
\end{tikzcd}$$

Furthermore, $\delta \circ \tau^{-1} \circ j'^* = \tau^{-1} \circ \delta\colon H^n(A \times I \cup X \times \dot I, X \times \dot I) \to H^n(X,A)$. Since $f'' = \bar f' \circ h_1$, then $f''^* = h_1^* \circ \bar f'^*$, and we have

$$(-1)^{n-1}\tau^{-1}\delta(\theta(f))^*(\iota) = \delta f''^*(\iota')$$

By definition, the left-hand side above equals $-c(f)$. ∎

3 MOORE-POSTNIKOV FACTORIZATIONS

This section is devoted to a method of factorizing a large class of maps up to weak homotopy type as infinite composites of simpler maps, the simpler maps being of the same weak homotopy type as principal fibrations of type (π,n) for some π and n. The cohomological description of the lifting problem for these fibrations, given in the last section, will lead us ultimately to an iterative attack on general lifting problems.

Given a sequence of fibrations $E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots$, we define

$$E_\infty = \lim_{\leftarrow} \{E_q,p_q\} = \{(e_q) \in \times E_q \mid p_q(e_q) = e_{q-1}\}$$

and we define $a_q: E_\infty \to E_q$ to be the projection of E_∞ to the qth coordinate. Then each map a_q is a fibration and $a_q = p_{q+1} \circ a_{q+1}$ for $q \geq 0$. For any space X a map $f: X \to E_\infty$ corresponds bijectively to a sequence of maps $\{f_q: X \to E_q\}_{q\geq 0}$ such that $f_q = p_{q+1} \circ f_{q+1}$ for $q \geq 0$ (given f, the sequence $\{f_q\}$ is defined by $f_q = a_q \circ f$). In particular, given a pair (X,A) with inclusion map $i: A \subset X$ and a map pair $f: i \to a_0$ consisting of the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f''} & E_\infty \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle a_0} \\ X & \xrightarrow{f'} & E_0 \end{array}$$

a lifting $\bar{f}: X \to E_\infty$ corresponds bijectively to a sequence of maps $\{\bar{f}_q: X \to E_q\}_{q\geq 0}$ such that

(a) $\bar{f}_0 = f': X \to E_0$

(b) For $q \geq 1$ the map $\bar{f}_q: X \to E_q$ is a lifting of the map pair from i to p_q consisting of the commutative square

$$\begin{array}{ccc} A & \xrightarrow{a_q \circ f''} & E_q \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_q} \\ X & \xrightarrow{\bar{f}_{q-1}} & E_{q-1} \end{array}$$

In this way the relative-lifting problem for a map pair $f: i \to a_0$ corresponds to a sequence of relative-lifting problems for map pairs from i to p_q. In many cases the relative-lifting problems for the fibrations p_q may be simpler to deal with than the original relative-lifting problem for the fibration a_0.

A sequence of fibrations $E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots$ is said to be *convergent* if for any $n < \infty$ there is N_n such that p_q is an n-equivalence for $q > N_n$.

Let $f: Y' \to Y$ be a map. A *convergent factorization* of f consists of a sequence $\{p_q,E_q,f_q\}_{q\geq 1}$ such that

(a) For $q > 1$, $p_q: E_q \to E_{q-1}$ is a fibration, and for $q = 1$, $p_1: E_1 \to Y$ is a fibration.

(b) For $q \geq 1$, $f_q: Y' \to E_q$ is a map, $f_q = p_{q+1} \circ f_{q+1}$ for $q \geq 1$, and $f = p_1 \circ f_1$.

(c) For any $n < \infty$ there is N_n such that f_q is an n-equivalence for $q > N_n$.

Conditions (a) and (b) imply that for $q \geq 1$, f equals the composite

$p_1 \circ \cdots \circ p_q \circ f_q$. The convergence condition (*c*) implies that, in a certain sense, the infinite composite $p_1 \circ p_2 \circ \cdots$ exists.

If $\{p_q,E_q,f_q\}_{q\geq 1}$ is a convergent factorization of a map $f: Y' \to Y$, then the sequence of fibrations $Y \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots$ is convergent. The following theorem shows that any convergent sequence of fibrations is obtained in this way from a convergent factorization of some map.

1 THEOREM *If $E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots$ is a convergent sequence of fibrations, then $\{p_q,E_q,a_q\}_{q\geq 1}$ is a convergent factorization of the map $a_0: E_\infty \to E_0$.*

PROOF Conditions (*a*) and (*b*) for a convergent factorization are clearly satisfied. To prove that the convergence condition (*c*) is also satisfied, given $1 \leq n < \infty$, choose N so that p_q is an $(n+1)$-equivalence if $q \geq N$. We prove that a_q is an n-equivalence for $q \geq N$. Because $a_q = p_{q+1} \circ a_{q+1}$, and p_{q+1} is an $(n+1)$-equivalence for $q \geq N$, it suffices to prove that a_N is an n-equivalence.

Let (P,Q) be a polyhedral pair such that $\dim P \leq n$ and let $\alpha: Q \to E_\infty$ and $\beta'_N: P \to E_N$ be maps such that $\beta'_N \mid Q = a_N \circ \alpha$. We now prove that there is an extension $\beta: P \to E_\infty$ of α such that $a_N \circ \beta = \beta'_N$. The map α corresponds to a sequence $\alpha_q = a_q \circ \alpha: Q \to E_q$ such that $\alpha_q = p_{q+1} \circ \alpha_{q+1}$, and to define a map $\beta: P \to E_\infty$ with the desired properties, we must obtain a sequence of maps $\beta_q: P \to E_q$ such that $\beta_q \mid Q = \alpha_q$, $\beta_q = p_{q+1} \circ \beta_{q+1}$, and $\beta_N = \beta'_N$. Such a sequence of maps $\{\beta_q\}$ is defined for $q \leq N$ by $\beta_q = p_{q+1} \circ \cdots \circ p_N \circ \beta'_N$, and for $q \geq N$ it is defined by induction on q as follows. Assuming β_q defined for some $q \geq N$, we use theorem 7.6.22 to find a map $\beta'_{q+1}: P \to E_{q+1}$ such that $\beta'_{q+1} \mid Q = \alpha_{q+1}$ and such that $\beta_q \simeq p_{q+1} \circ \beta'_{q+1}$ rel Q. We use the fact that p_{q+1} is a fibration (and theorem 7.2.6) to alter β'_{q+1} by a homotopy relative to Q to obtain a map $\beta_{q+1}: P \to E_{q+1}$ such that $\beta_{q+1} \mid Q = \alpha_{q+1}$ and such that $\beta_q = p_{q+1} \circ \beta_{q+1}$. Thus the sequence $\{\beta_q\}$ can be found, and hence a map $\beta: P \to E_\infty$ with the requisite properties exists.

Taking P to be a single point and Q to be empty, we see that a_N is surjective, and so a_N maps $\pi_0(E_\infty)$ surjectively to $\pi_0(E_N)$. Taking $(P,Q) = (I,\dot{I})$, we see that a_N maps $\pi_0(E_\infty)$ injectively to $\pi_0(E_N)$. Then a_N induces a one-to-one correspondence between the set of path components of E_∞ and the set of path components of E_N.

Let $e_* = (e_q) \in E_\infty$ be arbitrary and let $1 \leq k \leq n$. Taking $(P,Q) = (S^k,z_0)$ it follows that $a_{N\#}$ maps $\pi_k(E_\infty,e_*)$ epimorphically to $\pi_k(E_N,e_N)$. For $1 \leq k < n$, taking $(P,Q) = (E^{k+1},S^k)$, it follows that $a_{N\#}$ maps $\pi_k(E_\infty,e_*)$ monomorphically to $\pi_k(E_N,e_N)$. Hence a_N is an n-equivalence. ∎

2 COROLLARY *Let $\{p_q,E_q,f_q\}_{q\geq 1}$ be a convergent factorization of a map $f: Y' \to Y$ and let $f': Y' \to E_\infty$ be the map such that $a_q \circ f' = f_q$ for $q \geq 1$ and $a_0 \circ f' = f$. Then f' is a weak homotopy equivalence.*

PROOF For any $1 \leq n < \infty$ there is q such that a_q and f_q are both $(n+1)$-equivalences (by theorem 1). Then f' is also an n-equivalence (because $a_q \circ f' = f_q$). Since this is so for all n, f' is a weak homotopy equivalence. ∎

In particular, given a convergent factorization $\{p_q,E_q,f_q\}_{q\geq 1}$ of a weak fibration $p\colon E \to B$, there is a weak homotopy equivalence $g\colon p \to a_0$ consisting of the commutative square

$$\begin{array}{ccc} E & \xrightarrow{f'} & E_\infty \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle a_0} \\ B & \xrightarrow{1} & B \end{array}$$

If (X,A) is a relative CW complex, with inclusion map $i\colon A \subset X$, it follows from theorem 7.8.12 that the relative-lifting problem for a map pair $h\colon i \to p$ is equivalent to the relative lifting problem for the map pair $g \circ h\colon i \to a_0$. We shall now add hypotheses which will ensure that the sequence of fibrations into which the fibration a_0 is factored (namely, the fibrations $\{p_q\}$) leads to relative-lifting problems which can be settled by the methods of the last section.

A *Moore-Postnikov sequence* of fibrations $E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots$ is a convergent sequence of fibrations such that $p_q\colon E_q \to E_{q-1}$ is a principal fibration of type (G_q,n_q) for $q \geq 1$. A *Moore-Postnikov factorization* of a map $f\colon Y' \to Y$ is a convergent factorization $\{p_q,E_q,f_q\}_{q\geq 1}$ of f such that $E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots$ is a Moore-Postnikov sequence of fibrations. A *Postnikov factorization* of a space Y' is a Moore-Postnikov factorization of the map $f\colon Y' \to Y$, where Y is the set of path components of Y' topologized by the quotient topology and f is the collapsing map. Thus, if Y' is path connected, a Postnikov factorization of Y' is a Moore-Postnikov factorization of the constant map $Y' \to y_0$.

A Moore-Postnikov factorization of a map is a factorization of the map (up to weak homotopy type) as an infinite composite of elementary maps. The relative-lifting problem associated to this sequence is thereby factored into an infinite sequence of elementary relative-lifting problems. We shall show that Moore-Postnikov factorizations exist for a large class of maps between path-connected spaces.

Let $f\colon Y' \to Y$ be a map between path-connected pointed spaces. For $n \geq 1$ an *n-factorization of f* is a factorization of f as a composite $Y' \xrightarrow{b'} E' \xrightarrow{p'} Y$ such that

(*a*) E' is a path-connected pointed space, p' is a fibration, and b' is a lifting of f (that is, $f = p' \circ b'$)
(*b*) $b'_\#\colon \pi_q(Y') \to \pi_q(E')$ is an isomorphism for $1 \leq q < n$ and an epimorphism for $q = n$ (that is, b' is an n-equivalence)
(*c*) $p'_\#\colon \pi_q(E') \to \pi_q(Y)$ is an isomorphism for $q > n$ and a monomorphism for $q = n$

A map $f\colon Y' \to Y$ between path-connected pointed spaces is said to be *simple* if $f_\#(\pi_1(Y'))$ is a normal subgroup of $\pi_1(Y)$ and the quotient group is abelian, and if (Z_f,Y') is n-simple for $n \geq 1$ (as defined in Sec. 7.3). We are heading toward a proof of the result that a simple map admits Moore-Postnikov factorizations. We need one more auxiliary concept.

Given a pointed pair (X,A) of path-connected spaces, a cohomology class $v \in H^n(X,A;\, \pi)$ is said to be *n-characteristic* for (X,A) if either of the following conditions hold:

(*a*) $n = 1$ and $i_\#(\pi_1(A))$ is a normal subgroup of $\pi_1(X)$ whose quotient group is mapped isomorphically onto π by the composite

$$\pi_1(X)/i_\#(\pi_1(A)) \xrightarrow{\varphi} H_1(X)/i_*(H_1(A)) \xrightarrow{j^*} H_1(X,A) \xrightarrow{h(v)} \pi$$

(*b*) $n > 1$ and the composite

$$\pi_n(X,A) \xrightarrow{\varphi} H_n(X,A) \xrightarrow{h(v)} \pi$$

is an isomorphism

In case $A = \{x_0\}$, the concept of n-characteristic element for the pair $(X,\{x_0\})$ agrees with the concept of n-characteristic element for the space X as defined in Sec. 8.1.

3 LEMMA *Let $i\colon A \subset X$ be a simple inclusion map between path-connected pointed spaces such that the pair (X,A) is $(n-1)$-connected, where $n \geq 1$. Then there exist cohomology classes $v \in H^n(X,A;\, \pi)$ which are n-characteristic for (X,A), where $\pi = \pi_1(X)/i_\#(\pi_1(A))$ for $n = 1$* and *$\pi = \pi_n(X,A)$ for $n > 1$.*

PROOF If $n = 1$, it follows from the absolute Hurewicz isomorphism theorem applied to A and to X that there are isomorphisms

$$\pi_1(X)/i_\#(\pi_1(A)) \overset{\varphi}{\approx} H_1(X)/i_*(H_1(A)) \overset{j_*}{\approx} H_1(X,A)$$

By the universal-coefficient formula for cohomology, there is also an isomorphism

$$h\colon H^1(X,A;\, \pi) \approx \operatorname{Hom}(H_1(X,A), \pi)$$

Hence, if $\pi = \pi_1(X)/i_\#(\pi_1(A))$, there exist 1-characteristic elements $v \in H^1(X,A;\, \pi)$.

If $n > 1$, it follows from the relative Hurewicz isomorphism theorem and the universal-coefficient formula for cohomology that there are isomorphisms $\varphi\colon \pi_n(X,A) \approx H_n(X,A)$ and $h\colon H^n(X,A;\, \pi) \approx \operatorname{Hom}(H_n(X,A),\pi)$. Therefore, if $\pi = \pi_n(X,A)$, there are n-characteristic elements $v \in H^n(X,A;\, \pi)$. ∎

4 LEMMA *Let (X,A) be a pointed pair of path-connected spaces $(n-1)$-connected for some $n \geq 1$ and such that the inclusion map $i\colon A \subset X$ is simple. Then there is an n-factorization $A \xrightarrow{b'} E' \xrightarrow{p'} X$ of i such that p' is a principal fibration of type (π,n), where $\pi = \pi_1(X)/i_\#(\pi_1(A))$ if $n = 1$ and $\pi = \pi_n(X,A)$ if $n > 1$.*

PROOF By lemma 3, there is a class $v \in H^n(X,A;\, \pi)$ which is n-characteristic for (X,A). Let CA be the cone (nonreduced) over A and observe that $\{X,CA\}$ is an excisive couple in $X \cup CA$. Therefore there is an element $v' \in H^n(X \cup CA;\, \pi)$ corresponding to v under the isomorphisms

$$H^n(X \cup CA;\, \pi) \underset{\approx}{\leftarrow} H^n(X \cup CA, CA;\, \pi) \underset{\approx}{\rightarrow} H^n(X,A;\, \pi)$$

It is possible to imbed $X \cup CA$ in a space X' of type (π,n) having an n-characteristic element ι' such that $\iota' \mid X \cup CA = v'$. Let p': $E' \to X$ be the principal fibration induced by the inclusion $X \subset X'$ and let p_A': $E_A' \to A$ be the restriction of this fibration to A. There is a section s: $A \to E_A'$ such that $s(a) = (a,\omega_a)$ for $a \in A$, where ω_a is the path from x_0 to the vertex of CA followed by the path from the vertex of CA to a (that is, $\omega_a(t) = [x_0, 1 - 2t]$ for $0 \leq t \leq ½$ and $\omega_a(t) = [a, 2t - 1]$ for $½ \leq t \leq 1$). We define b': $A \to E'$ to be the composite $A \xrightarrow{s} E_A' \overset{i_A}{\subset} E'$ and shall prove that $A \xrightarrow{b'} E' \xrightarrow{p'} X$ is an n-factorization of i.

The fiber of p' (and hence also of p_A') is $\Omega X'$, and we define g: $E_A' \to \Omega X'$ by $g(a,\omega) = \omega * (s(a))^{-1}$. Then $g \mid \Omega X'$: $\Omega X' \to \Omega X'$ is homotopic to the identity map. If i'': $\Omega X' \subset E_A'$ is the inclusion map, it follows from the exactness of the homotopy sequence of the fibration p_A': $E_A' \to A$ that there is a direct-sum decomposition

$$\pi_q(E_A') \approx i''_\# \pi_q(\Omega X') \oplus s_\# \pi_q(A) \qquad q \geq 1$$

(This is a direct-product decomposition for $q = 1$, but we shall still write it additively.) We define a homomorphism λ: $\pi_q(X,A) \to \pi_{q-1}(\Omega X')$, where $q \geq 1$, to be the composite

$$\pi_q(X,A) \xrightarrow[\approx]{p_\#'^{-1}} \pi_q(E',E_A') \xrightarrow{\partial} \pi_{q-1}(E_A') \xrightarrow{g_\#} \pi_{q-1}(\Omega X')$$

We show that the following diagram commutes up to sign:

$$\begin{tikzcd}
\pi_q(A) \arrow[r, "i_\#"] \arrow[d, "b_\#'"'] & \pi_q(X) \arrow[r, "j_\#"] \arrow[d, "="] & \pi_q(X,A) \arrow[r, "\partial"] \arrow[d, "\lambda"] & \pi_{q-1}(A) \arrow[d, "b_\#'"] \\
\pi_q(E') \arrow[r, "p_\#'"] & \pi_q(X) \arrow[r, "\bar\partial"] & \pi_{q-1}(\Omega X') \arrow[r, "i_\#'"] & \pi_{q-1}(E')
\end{tikzcd}$$

In fact, the left-hand and middle squares are easily seen to be commutative. We shall show that $b_\#' \circ \partial = -i_\#' \circ \lambda$.

For $q = 1$ this is so because $\pi_0(A) = 0$ implies that $b_\#' \circ \partial$ is the trivial map and the fact that $j_\#$ is surjective and $i_\#' \circ \lambda \circ j_\# = i_\#' \circ \bar\partial = 0$ implies that $i_\#' \circ \lambda$ is also the trivial map. For $q > 1$ we have

$$\alpha = i''_\# g_\# \alpha + s_\# p_{A\#}' \alpha \qquad \alpha \in \pi_{q-1}(E_A')$$

Since the composite $\pi_q(E',E_A') \xrightarrow{\partial} \pi_{q-1}(E_A') \xrightarrow{i_{A\#}'} \pi_{q-1}(E')$ is trivial, it follows that for $\beta \in \pi_q(E',E_A')$

$$\begin{aligned} 0 = i_{A\#}' \partial\beta &= i_{A\#}' i''_\# g_\# \partial\beta + i_{A\#}' s_\# p_{A\#}' \partial\beta \\ &= i_\#' g_\# \partial\beta + b_\#' \partial p_\# \beta \end{aligned}$$

By definition of λ, we see that $\lambda p_\# \beta = g_\# \partial\beta$. Therefore

$$i_\#' \lambda p_\# \beta + b_\#' \partial p_\# \beta = 0$$

Since $p_\#$: $\pi_q(E',E_A') \approx \pi_q(X,A)$, this proves $b_\#' \circ \partial = -i_\#' \circ \lambda$.

A straightforward verification shows that λ is also the composite

$$\pi_n(X,A) \to \pi_n(X \cup CA,\ CA) \underset{\approx}{\leftarrow} \pi_n(X \cup CA) \to \pi_n(X') \underset{\approx}{\overset{\partial}{\to}} \pi_{n-1}(\Omega X')$$

The construction of X' and $\iota' \in H^n(X',\pi)$ shows that there is a commutative diagram

$$\begin{tikzcd}
\pi_n(X,A) \arrow[r] \arrow[d, "\varphi"', "\approx"] & \pi_n(X \cup CA,\ CA) \arrow[d, "\varphi"'] & \pi_n(X \cup CA) \arrow[l, "\approx"'] \arrow[r] \arrow[d, "\varphi"'] & \pi_n(X') \arrow[d, "\approx"', "\varphi"] \\
H_n(X,A) \arrow[r] \arrow[drr, "\approx", "h(\upsilon)"'] & H_n(X \cup CA,\ CA) \arrow[dr] & H_n(X \cup CA) \arrow[l, "\approx"'] \arrow[r] \arrow[d, "h(\upsilon')"] & H_n(X') \arrow[dl, "\approx"', "h(\iota')"] \\
 & & \pi &
\end{tikzcd}$$

Therefore λ: $\pi_n(X,A) \approx \pi_{n-1}(\Omega X')$.

In case $n = 1$, ∂: $\pi_1(X) \to \pi_0(\Omega X')$ is surjective [because $\pi_0(A) = 0$], and so E' is path connected. If $n > 1$, E' is path connected because $\pi_0(\Omega X') = 0$. Therefore E' is a path-connected pointed space. Since $\pi_q(\Omega X') = 0$ for $q \geq n$, it follows from the exactness of the homotopy sequence of the fibration p': $E' \to X$ that $p'_\#$: $\pi_q(E') \to \pi_q(X)$ is an isomorphism for $q > n$ and a monomorphism for $q = n$.

Because λ: $\pi_q(X,A) \to \pi_{q-1}(\Omega X')$ is a bijection for $q \leq n$ (the only nontrivial case in these dimensions being $q = n$), it follows from the five lemma and the commutativity up to sign of the diagram on page 442 that $b'_\#$: $\pi_q(A) \to \pi_q(E')$ is an isomorphism for $1 \leq q < n$ and an epimorphism for $q = n$. Therefore b' and p' have the properties required of an n-factorization of i. ∎

5 COROLLARY *Let g: $X' \to X$ be a simple map between path-connected pointed spaces such that for some $n \geq 1$ the map $g_\#$: $\pi_q(X') \to \pi_q(X)$ is an isomorphism for $1 \leq q < n - 1$ and an epimorphism for $q = n - 1$. Then there is an n-factorization $X' \xrightarrow{b'} E' \xrightarrow{p'} X$ of g such that p' is a principal fibration of type (π,n) for some abelian group π.*

PROOF Let Z be the reduced mapping cylinder of g (that is, the mapping cylinder of $g \mid x_0'$: $x_0' \to x_0$ has been collapsed to a point). Then (Z,X') is a pointed pair of path-connected spaces $(n - 1)$-connected and with simple inclusion map i: $X' \subset Z$. By lemma 4, there is an n-factorization $X' \xrightarrow{b''} E'' \xrightarrow{p''} Z$ of i such that p'' is a principal fibration of type (π,n). Let p': $E' \to X$ be the restriction of p'' to X. Then $E' \subset E''$ is a homotopy equivalence, so there is a map $\bar{b}''$: $X' \to E'$ such that b'' is homotopic to the composite $X' \xrightarrow{\bar{b}''} E' \subset E''$. Then $p' \circ \bar{b}''$ is easily seen to be homotopic to g. By the homotopy lifting property of p', there is a map b': $X' \to E'$ homotopic to $\bar{b}''$ such that $p' \circ b' = g$. Then $X' \xrightarrow{b'} E' \xrightarrow{p'} X$ is easily verified to have the requisite properties. ∎

We are now ready to prove the existence of Moore-Postnikov factorizations of a simple map between path-connected pointed spaces.

6 THEOREM *Let $f: Y' \to Y$ be a simple map between path-connected pointed spaces. There is a Moore-Postnikov factorization $\{p_q,E_q,f_q\}_{q\geq 1}$ of f such that for $n \geq 1$ the sequence*

$$Y' \xrightarrow{f_n} E_n \xrightarrow{p_1 \circ \cdots \circ p_n} Y$$

is an n-factorization of f.

PROOF By induction on q, we prove the existence of a sequence $\{p_q,E_q,f_q\}_{q\geq 1}$ such that

(*a*) For $n = 1$ the sequence $Y' \xrightarrow{f_1} E_1 \xrightarrow{p_1} Y$ is a 1-factorization of f.

(*b*) For $n > 1$ the sequence $Y' \xrightarrow{f_n} E_n \xrightarrow{p_n} E_{n-1}$ is an n-factorization of f_{n-1}.

(*c*) For $n \geq 1$, p_n is a principal fibration of type (π_n,n) for some π_n.

Once such a sequence $\{p_q,E_q,f_q\}$ has been found, it is easy to verify that it is a Moore-Postnikov factorization of f with the desired property. Therefore we limit ourselves to proving the existence of such a sequence.

By corollary 5, with $n = 1$, there is a 1-factorization $Y' \xrightarrow{f_1} E_1 \xrightarrow{p_1} Y$ of f with p_1 a principal fibration of type $(\pi_1,1)$ for some π_1. This defines p_1, E_1, and f_1. Assume $\{p_q,E_q,f_q\}$ defined for $1 \leq q < n$, where $n > 1$, to satisfy (*a*), (*b*), and (*c*) above. By corollary 5, there is an n-factorization $Y' \xrightarrow{f_n} E_n \xrightarrow{p_n} E_{n-1}$ of f_{n-1} such that p_n is a principal fibration of type (π_n,n) for some π_n. Then p_n, E_n, and f_n have the desired properties. ∎

7 COROLLARY *Let Y' be a simple path-connected pointed space. Then Y' has a Postnikov factorization $\{p_q,E_q,f_q\}_{q\geq 1}$ in which $\pi_q(E_n) = 0$ for $q \geq n$ and $f_n: Y' \to E_n$ is an n-equivalence.*

PROOF If Y' is a simple space, the constant map $Y' \to y_0$ is a simple map. The result follows from theorem 6. ∎

In the above the spaces E_n approximate Y' in low dimensions. We now present an alternate method of approximating a space in high dimensions by killing low-dimensional homotopy groups.

8 COROLLARY *Let Y be a simple path-connected pointed space. There is a Moore-Postnikov sequence of fibrations $Y \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots$ such that E_n is n-connected and $p_1 \circ \cdots \circ p_n: E_n \to Y$ induces isomorphisms $\pi_q(E_n) \approx \pi_q(Y)$ for $q > n$.*

PROOF If Y is a simple space, the inclusion map $y_0 \subset Y$ is a simple map. The result then follows from theorem 6. ∎

In the last result the fibration $p_1: E_1 \to Y$ has the homotopy properties of a universal covering space of Y. The fibration $p_1 \circ \cdots \circ p_n: E_n \to Y$ is a kind of "n-covering space."

4 OBSTRUCTION THEORY

In this section we show how to use Moore-Postnikov factorizations to study the relative-lifting problem. A sequence of obstructions to the existence of a lifting (or to the existence of a homotopy between two liftings) is defined iteratively, and we apply the general machinery to the special case where either the first one or the first two obstructions are the only ones that enter.

Let $p: E \to B$ be a fibration between path-connected pointed spaces and assume that p is a simple map. By theorem 8.3.6, there exist Moore-Postnikov factorizations $\{p_q,E_q,f_q\}_{q\geq 1}$ of p. By corollary 8.3.2, there is a map $p': E \to E_\infty$ which is a weak homotopy equivalence. Since $p = a_0 \circ p'$, where $a_0: E_\infty \to B$, if (X,A) is a relative CW complex, with $i: A \subset X$, it follows from theorem 7.8.12 that the relative-lifting problem for a map pair from i to p is equivalent to the relative-lifting problem for a corresponding map pair from i to a_0. Thus we are led to consider the relative-lifting problem for a map pair from i to a_0.

Let $E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots$ be a sequence of fibrations with limit E_∞ and maps $a_q: E_\infty \to E_q$ and let (X,A) be a relative CW complex, with inclusion map $i: A \subset X$. A map pair $f: i \to a_0$ is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f''} & E_\infty \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle a_0} \\ X & \xrightarrow{f'} & E_0 \end{array}$$

where f'' corresponds to a collection $\{f_q'': A \to E_q\}_{q\geq 0}$ such that $p_{q+1} \circ f_{q+1}'' = f_q''$ for $q \geq 0$. For $q \geq 1$ let $f_q: i \to p_1 \circ \cdots \circ p_q$ be the map pair consisting of the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f_q''} & E_q \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_1 \circ \cdots \circ p_q} \\ X & \xrightarrow{f'} & E_0 \end{array}$$

If $\bar{f}_q: X \to E_q$ is a lifting of f_q, then $p_q \circ \bar{f}_q$ is a lifting of f_{q-1} for $q > 1$ and a lifting $\bar{f}: X \to E_\infty$ of f corresponds to a sequence $\{\bar{f}_q: X \to E_q\}_{q\geq 1}$ such that

(a) $\bar{f}_q$ is a lifting of f_q for $q \geq 1$.
(b) $p_{q+1} \circ \bar{f}_{q+1} = \bar{f}_q$ for $q \geq 1$.

Given a lifting $\bar{f}_q: X \to E_q$ of f_q for $q \geq 1$, let $g(\bar{f}_q): i \to p_{q+1}$ be the map pair consisting of the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f_{q+1}''} & E_{q+1} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_{q+1}} \\ X & \xrightarrow{\bar{f}_q} & E_q \end{array}$$

A map $\bar{f}_{q+1}\colon X \to E_{q+1}$ is a lifting of $g(\bar{f}_q)$ if and only if it is a lifting of f_{q+1} such that $p_{q+1} \circ \bar{f}_{q+1} = \bar{f}_q$. Thus a sequence of maps $\{\bar{f}_q\colon X \to E_q\}_{q\geq 1}$ satisfies conditions (*a*) and (*b*) above if and only if it has the following properties:

(*c*) $\bar{f}_1$ is a lifting of f_1.
(*d*) For $q \geq 1$, $\bar{f}_{q+1}$ is a lifting of $g(\bar{f}_q)$.

We now add the hypothesis that $E_0 \xleftarrow{p_1} E_1 \xleftarrow{p_2} \cdots$ is a Moore-Postnikov sequence of fibrations. For each $q \geq 1$, p_q is then a principal fibration of type (π_q,n_q). It follows from Sec. 8.2 that f_1 can be lifted if and only if $c(f_1) \in H^{n_1}(X,A;\ \pi_1)$ is zero. The class $c(f_1)$ is called the *first obstruction to lifting* f.

Assume that for some $q > 1$ there exist liftings $\bar{f}_{q-1}\colon X \to E_{q-1}$ of the map pair $f_{q-1}\colon i \to p_1 \circ \cdots \circ p_{q-1}$. We then obtain map pairs $g(\bar{f}_{q-1})\colon i \to p_q$ and corresponding elements $c(g(\bar{f}_{q-1})) \in H^{n_q}(X,A;\ \pi_q)$. The collection $\{c(g(\bar{f}_{q-1}))\}$ corresponding to the set of all liftings $\bar{f}_{q-1}\colon X \to E_{q-1}$ of f_{q-1} is called the *qth obstruction to lifting* f. It is a subset of $H^{n_q}(X,A;\ \pi_q)$ and is defined if and only if f_{q-1} can be lifted. It is clear that there is a lifting of f_q if and only if the qth obstruction to lifting f is defined and contains the zero element of $H^{n_q}(X,A;\ \pi_q)$.

Corresponding to a Moore-Postnikov sequence of fibrations we have been led to a sequence of successive obstructions. The first obstruction is a single cohomology class, while the higher obstructions are subsets of cohomology groups. In some cases these obstructions can be effectively computed in terms of the given map pair $f\colon i \to a_0$, and this computation provides a solution of the lifting problem in these cases. In general, however, the determination of the successive obstructions involves an iterative procedure of increasing complexity and has not been effectively carried out in each case.

We illustrate this technique by applying it to the Postnikov factorization of a simple path-connected pointed space Y, given in corollary 8.3.7. There is a Postnikov factorization $\{p_q,E_q,f_q\}_{q\geq 1}$ of Y in which $\pi_q(E_m) = 0$ for $q \geq m$ and $f_m\colon Y \to E_m$ is an m-equivalence. We call this the *standard Postnikov factorization* of Y. By corollary 8.3.2, there is a weak homotopy equivalence $f'\colon Y \to E_\infty$, and so we consider the lifting problem for a map $i \to a_0$, where $i\colon A \subset X$ and $a_0\colon E_\infty \to y_0$. Since y_0 is a point, this is equivalent to the extension problem for a map $f''\colon A \to E_\infty$.

Thus we seek a sequence of maps $\bar{f}_q\colon X \to E_q$ such that $\bar{f}_1\colon X \to E_1$ is an extension of $a_1 \circ f''$ and $\bar{f}_{q+1}\colon X \to E_{q+1}$ for $q \geq 1$ is a lifting of the map pair $g(\bar{f}_q)\colon i \to p_{q+1}$ consisting of

$$\begin{array}{ccc} A & \xrightarrow{a_{q+1}\circ f''} & E_{q+1} \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p_{q+1}} \\ X & \xrightarrow{\bar{f}_q} & E_q \end{array}$$

Since p_{q+1} is a principal fibration of type $(\pi_q(Y,y_0),\ q+1)$, the obstruction to lifting $g(\bar{f}_q)$ is an element of $H^{q+1}(X,A;\ \pi_q(Y,y_0))$. Hence there is defined a sequence of obstructions to extending $f'': A \to Y$, the $(q+1)$st obstruction being a subset of $H^{q+1}(X,A;\ \pi_q(Y,y_0))$. If Y is $(n-1)$-connected for some $n \geq 1$, the lowest-dimensional nontrivial obstruction is in $H^{n+1}(X,A;\ \pi_n(Y,y_0))$. If $\iota \in H^n(Y,y_0;\ \pi)$ is n-characteristic for such a space Y, it follows easily from theorem 8.2.6 that this lowest obstruction is $\pm\delta f''^*\iota$. This gives us the following generalization of theorem 8.1.17.[1]

1 THEOREM *Let $\iota \in H^n(Y,y_0;\ \pi)$ be n-characteristic for a simple $(n-1)$-connected pointed space Y, where $n \geq 1$, and let (X,A) be a relative CW complex such that $H^{q+1}(X,A;\ \pi_q(Y,y_0)) = 0$ for $q > n$. A map $f\colon A \to Y$ can be extended over X if and only if $\delta f^*(\iota) = 0$ in $H^{n+1}(X,A;\ \pi)$.*

PROOF We use the standard Postnikov factorization of Y. This leads to a sequence of obstructions to extending f which are subsets of $H^{q+1}(X,A;\ \pi_q(Y,y_0))$. Since these are all zero except $H^{n+1}(X,A;\ \pi_n(Y,y_0)) \approx H^{n+1}(X,A;\ \pi)$, the only obstruction to extending f is an element of $H^{n+1}(X,A;\ \pi)$. By the remarks above, this obstruction vanishes if and only if $\delta f^*(\iota) = 0$. ■

Let $f_0, f_1\colon X \to Y$ be maps and define $g\colon X \times \dot{I} \to Y$ by $g(x,0) = f_0(x)$ and $g(x,1) = f_1(x)$. For any $u \in H^q(Y)$, $\delta g^*(u) = (-1)^q\tau(f_1^* u - f_0^* u)$ in $H^{q+1}(X \times I, X \times \dot{I})$. Therefore $\delta g^*(u) = 0$ if and only if $f_0^*(u) = f_1^*(u)$, and we obtain the following partial generalization of theorem 8.1.15 by applying theorem 1 to the pair $(X \times I, X \times \dot{I})$.

2 THEOREM *Let $\iota \in H^n(Y,y_0;\ \pi)$ be n-characteristic for a simple $(n-1)$-connected space Y, where $n \geq 1$, and let X be a CW complex such that $H^q(X;\ \pi_q(Y,y_0)) = 0$ for $q > n$. Then $f_0, f_1\colon X \to Y$ are homotopic if and only if $f_0^*(\iota) = f_1^*(\iota)$.* ■

This last result gives a condition that the map $\psi_\iota\colon [X;Y] \to H^n(X,\pi)$ be injective. The condition that ψ_ι be surjective is that if $\{p_q,E_q,f_q\}_{q\geq 1}$ is the standard Postnikov factorization of Y, then any map $X \to E_{n+1}$ can be lifted. The obstructions to lifting such a map lie in $H^{q+1}(X;\ \pi_q(Y,y_0))$ for $q > n$. Therefore, by combining these, we have the following result.

3 THEOREM *Let $\iota \in H^n(Y,y_0;\ \pi)$ be n-characteristic for a simple $(n-1)$-connected space Y, where $n \geq 1$, and let X be a CW complex such that $H^q(X;\pi_q(Y)) = 0$ and $H^{q+1}(X;\pi_q(Y)) = 0$ for all $q > n$. Then there is a bijection*

$$\psi_\iota\colon [X;Y] \approx H^n(X;\pi) \quad ■$$

These last results have been derived by assuming hypotheses which ensure that the lowest-dimensional obstruction is the only nontrivial one. In this case we are essentially studying maps to a space of type (π,n). The case where the two lowest-dimensional obstructions are the only nontrivial obstructions is essentially the study of maps to a fibration $E \to B$ of type (G,q), where B is a

[1] See S. Eilenberg, Cohomology and continuous mappings, *Annals of Mathematics*, vol. 41, pp. 231–251, 1940.

space of type (π,n). Before we consider this, let us establish some cohomology properties of $X \times I$.

Define inclusion maps

$$A \times I \cup X \times 1 \overset{i_1}{\subset} A \times I \cup X \times \dot{I} \overset{j_1}{\subset} (A \times I \cup X \times \dot{I}, A \times I \cup X \times 1)$$

There is a weak retraction $r: A \times I \cup X \times \dot{I} \to A \times I \cup X \times 1$ defined by $r(x,t) = (x,1)$ for $(x,t) \in A \times I \cup X \times \dot{I}$ (that is, $r \circ i_1$ is homotopic to the identity map of $A \times I \cup X \times 1$). Using the exactness of the cohomology sequence of $(A \times I \cup X \times \dot{I}, A \times I \cup X \times 1)$, it follows that for an arbitrary element $u \in H^q(A \times I \cup X \times \dot{I})$ there is an associated unique element $u' \in H^q(A \times I \cup X \times \dot{I}, A \times I \cup X \times 1)$ such that

$$u = j_1^* u' + r^* i_1^* u$$

Let $h: (X,A) \to (A \times I \cup X \times \dot{I}, A \times I \cup X \times 1)$ be defined by $h(x) = (x,0)$ for $x \in X$. Then h induces an isomorphism

$$h^*: H^q(A \times I \cup X \times \dot{I}, A \times I \cup X \times 1) \approx H^q(X,A)$$

and we define an epimorphism

$$\Delta: H^q(A \times I \cup X \times \dot{I}) \to H^q(X,A)$$

by $\Delta(u) = h^* u'$, where $u' \in H^q(A \times I \cup X \times \dot{I}, A \times I \cup X \times 1)$ is the unique element associated to u. Then Δ is a natural transformation on the category of pairs (X,A).

4 LEMMA *Commutativity holds in the triangle*

$$\begin{array}{ccc} H^q(A \times I \cup X \times \dot{I}) & \xrightarrow{\delta} & H^{q+1}((X,A) \times (I,\dot{I})) \\ & {\scriptstyle\Delta}\searrow \quad \nearrow{\scriptstyle(-1)^{q+1}\tau} & \\ & H^q(X,A) & \end{array}$$

PROOF Let $\bar{r}: X \times I \to A \times I \cup X \times 1$ be defined by $\bar{r}(x,t) = (x,1)$. Then $\bar{r} \mid (A \times I \cup X \times \dot{I}) = r$, and so $r^* i_1^* u = (\bar{r}^* i_1^* u) \mid (A \times I \cup X \times \dot{I})$ for $u \in H^q(A \times I \cup X \times \dot{I})$. For any $v \in H^q(X \times I)$, $\delta(v \mid (A \times I \cup X \times \dot{I})) = 0$. Therefore, $\delta r^* i_1^* u = 0$, and to complete the proof it suffices to show that for $u' \in H^q(A \times I \cup X \times \dot{I}, A \times I \cup X \times 1)$, $\delta j_1^*(u') = (-1)^{q+1}\tau h^*(u')$. This follows from the commutativity of a diagram analogous to the one used in the proof of theorem 8.2.4. ■

5 COROLLARY *Let (X,A) be a relative CW complex, with inclusion map $i: A \subset X$, and let $p': \Omega B' \to b_0'$ be the constant map, where B' is a space of type $(\pi,n+1)$. Given a map pair $f: i \to p'$ and two liftings $f_0, f_1: X \to \Omega B'$ of f, let $g'': A \times I \cup X \times \dot{I} \to \Omega B'$ be defined by $g''(x,0) = f_0(x)$, $g''(x,1) = f_1(x)$, and $g''(a,t) = f_0(a)$. If $\iota' \in H^n(\Omega B',\omega_0'; \pi)$ and $\iota \in H^{n+1}(B',b_0'; \pi)$ are related characteristic elements, then $d(f_0,f_1) = -\Delta g''^*(\iota')$.*

PROOF Let $g\colon i' \to p'$ be the map pair consisting of the commutative square

$$\begin{array}{ccc} A \times I \cup X \times \dot{I} & \xrightarrow{g''} & \Omega B' \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle p'} \\ X \times I & \xrightarrow{g'} & b_0' \end{array}$$

From the definition of $d(f_0,f_1)$ we have $d(f_0,f_1) = (-1)^{n+1}\tau^{-1}(c(g))$. By theorem 8.2.6 $c(g) = -\delta g''^*(\iota')$, and therefore $d(f_0,f_1) = (-1)^n\tau^{-1}\delta g''^*(\iota')$. The result follows from this and lemma 4. ∎

6 LEMMA *Let h_0, h_1: $(X,A) \to (A \times I \cup X \times \dot{I}, A \times I)$ be defined by $h_0(x) = (x,0)$ and $h_1(x) = (x,1)$. For any $u \in H^q(A \times I \cup X \times \dot{I}, A \times I)$*

$$\Delta(u \mid (A \times I \cup X \times \dot{I})) = h_0^*(u) - h_1^*(u)$$

PROOF There are inclusion maps

$$(A \times I \cup X \times 1, A \times I) \overset{i_1'}{\subset} (A \times I \cup X \times \dot{I}, A \times I) \overset{j_1'}{\subset}$$
$$(A \times I \cup X \times \dot{I}, A \times I \cup X \times 1)$$

and a weak retraction $r'\colon (A \times I \cup X \times \dot{I}, A \times I) \to (A \times I \cup X \times 1, A \times I)$ defined by $r'(x,t) = (x,1)$. For $v \in H^q(A \times I \cup X \times \dot{I}, A \times I)$ there is an associated unique element $v' \in H^q(A \times I \cup X \times \dot{I}, A \times I \cup X \times 1)$ such that

$$v = j_1'^* v' + r'^* i_1'^* v$$

If $k\colon A \times I \cup X \times \dot{I} \subset (A \times I \cup X \times \dot{I}, A \times I)$, we then have

$$k^* v = k^* j_1'^* v' + k^* r'^* i_1'^* v = j_1^* v' + r^* i_1^* k^* v$$

Therefore $\Delta k^* v = h^* v'$. Since $h = j_1' \circ h_0$ and $h_1 = i_1' \circ r' \circ h_0$, we have

$$\Delta k^* v = h_0^* j_1'^* v' = h_0^*(v - r'^* i_1'^* v) = h_0^* v - h_1^* v \quad ∎$$

7 COROLLARY *Given a map pair $g\colon i' \to p$, where (X,A) is a relative CW complex, $i'\colon A \times I \subset A \times I \cup X \times \dot{I}$, and $p\colon E \to B$ is a principal fibration of type (G,q) induced by a map $\theta\colon B \to B'$, let f_0, f_1: $i \to p$ be the map pairs from $i\colon A \subset X$ to p defined by restriction of g to $(X,A) \times 0$ and $(X,A) \times 1$, respectively. Then*

$$\Delta g'^* \theta^*(\iota) = c(f_0) - c(f_1)$$

where $g'\colon A \times I \cup X \times \dot{I} \to B$ is part of the map pair g.

PROOF The obstruction $c(g) \in H^q(A \times I \cup X \times \dot{I}, A \times I; G)$ has the property that $c(g) \mid (A \times I \cup X \times \dot{I})$ is the obstruction to lifting g'. Therefore

$$c(g) \mid (A \times I \cup X \times \dot{I}) = g'^* \theta^*(\iota)$$

By the naturality of the obstruction, $h_0^* c(g) = c(f_0)$ and $h_1^* c(g) = c(f_1)$. The result now follows from lemma 6. ∎

Let θ be a cohomology operation of type $(n,q;\ \pi,G)$. Given a cohomology class $u \in H^n(X;\pi)$, we define a map $\Delta(\theta,u)\colon H^n(X,A;\ \pi) \to H^q(X,A;\ G)$ by

$$\Delta(\theta,u)(v) = \Delta\theta(j_1^* h^{*-1}(v) + k^* u) \qquad v \in H^n(X,A;\ \pi)$$

where $k\colon A \times I \cup X \times \dot{I} \to X$ is defined by $k(x,t) = x$. In case θ is an additive cohomology operation, we have

$$\Delta(\theta,u)(v) = \Delta(j_1^* h^{*-1}\theta(v) + k^*\theta(u)) = \theta(v)$$

Therefore $\Delta(\theta,u) = \theta$ if θ is additive.

Given a cohomology operation θ of type $(n,q;\ \pi,G)$ and a cohomology class $u \in H^n(X;\pi)$, we define a map $S\Delta(\theta,u)\colon H^{n-1}(X,A;\ \pi) \to H^{q-1}(X,A;\ G)$ by the equation $S\Delta(\theta,u) = \tau^{-1} \circ \Delta(\theta,u') \circ \tau$, where $u' \in H^n(X \times I;\ \pi)$ is the image of u under the homomorphism induced by the projection $X \times I \to X$. If θ is an additive operation, then $S\Delta(\theta,u) = S\theta$. In any case, we have the following analogue of corollary 8.1.14.

8 LEMMA *If θ is a cohomology operation of type $(n,q;\ \pi,G)$ and $u \in H^n(X;\pi)$, the map*

$$S\Delta(\theta,u)\colon H^{n-1}(X,A;\ \pi) \to H^{q-1}(X,A;\ G)$$

is a homomorphism.

PROOF Let $I_1 = [0,\frac{1}{2}]$, $\dot{I}_1 = \{0,\frac{1}{2}\}$, $I_2 = [\frac{1}{2},1]$, and $\dot{I}_2 = \{\frac{1}{2},1\}$, and let $v_1,\ v_2 \in H^{n-1}(X,A;\ \pi)$. Let $v_1' = \tau_1(v_1) \in H^n((X,A) \times (I_1,\dot{I}_1))$ and let $v_2' = \tau_2(v_2) \in H^n((X,A) \times (I_2,\dot{I}_2))$, and let $v \in H^n((X,A) \times (I,\ \dot{I}_1 \cup \dot{I}_2))$ be the unique class such that $v \mid (X,A) \times (I_1,\dot{I}_1) = v_1'$ and $v \mid (X,A) \times (I_2,\dot{I}_2) = v_2'$. Then $v \mid (X,A) \times (I,\dot{I}) = \tau(v_1) + \tau(v_2)$. Since θ and Δ are both natural,

$$\Delta(\theta,u')(v) \mid (X,A) \times (I,\dot{I}) = \tau S\Delta(\theta,u)(v_1 + v_2)$$

and

$$\Delta(\theta,u')(v) \mid (X,A) \times (I_1,\dot{I}_1) = \tau_1 S\Delta(\theta,u)(v_1)$$
$$\Delta(\theta,u')(v) \mid (X,A) \times (I_2,\dot{I}_2) = \tau_2 S\Delta(\theta,u)(v_2)$$

Therefore, as in the proof of lemma 8.2.3,

$$\tau S\Delta(\theta,u)(v_1 + v_2) = \tau S\Delta(\theta,u)(v_1) + \tau S\Delta(\theta,u)(v_2)$$

Since τ is an isomorphism, this gives the result. ∎

Let B be a space of type (π,n) and let $p\colon E \to B$ be a principal fibration of type (G,q) induced by a map $\bar{\theta}\colon B \to B'$. Let $\theta' = \bar{\theta}^*(\iota') \in H^q(B,b_0;\ G)$ correspond to a cohomology operation θ of type $(n,q;\ \pi,G)$ (that is, $\theta(\iota) = \theta'$). Given a CW complex X, a map $f\colon X \to B$ can be lifted to E if and only if $\theta(f^*(\iota)) = 0$. For any element $u \in H^n(X;\pi)$ such that $\theta(u) = 0$ it follows that there are liftings $f\colon X \to E$ such that $(p \circ f)^*(\iota) = u$. We shall determine how many homotopy classes of such liftings there are.

9 LEMMA *Let $f_0,\ f_1\colon X \to E$ be maps such that $p \circ f_0 = p \circ f_1$ (that is, f_0 and f_1 are liftings of the same map $X \to B$). Then $f_0 \simeq f_1$ if and only if there is $d \in H^{n-1}(X;\pi)$ such that $d(f_0,f_1) = S\Delta(\theta,u)(d)$, where $u = (p \circ f_0)^*(\iota)$.*

PROOF Let $F_0: i' \to p$ be the map pair consisting of

$$\begin{array}{ccc} X \times \dot{I} & \xrightarrow{F_0''} & E \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{F_0'} & B \end{array}$$

where $F_0''(x,0) = f_0(x)$, $F_0''(x,1) = f_1(x)$, and $F_0'(x,t) = pf_0(x)$. Then $d(f_0,f_1) = (-1)^q\tau^{-1}(c(F_0))$. It is clear that $f_0 \simeq f_1$ if and only if there is a homotopy F_1': $X \times I \to B$ from $p \circ f_0$ to $p \circ f_1$ such that for the corresponding map pair F_1: $i' \to p$ we have $c(F_1) = 0$. Let G': $(X \times \dot{I}) \times I \cup (X \times I) \times \dot{I} \to B$ be defined by $G'(x,0,t) = G'(x,1,t) = pf_0(x)$, $G'(x,t,0) = F_0'(x,t)$ and $G'(x,t,1) = F_1'(x,t)$. By corollary 7,

$$\Delta G'^*(\theta') = c(F_0) - c(F_1)$$

Thus $f_0 \simeq f_1$ if and only if there is a map F_1': $X \times I \to B$ such that for the corresponding map G' we have

$$d(f_0,f_1) = (-1)^q\tau^{-1}(\Delta G'^*(\theta'))$$

It is easily verified that $G'^*(\iota) = j_1^* h^{*-1}\Delta G'^*(\iota) + k^*u'$, where $u' \in H^n(X \times I;\ \pi)$ is the image of $u = (p \circ f_0)^*(\iota)$ under the projection $X \times I \to X$. By definition,

$$\Delta G'^*(\theta') = \Delta G'^*\theta(\iota) = \Delta\theta G'^*(\iota) = \Delta(\theta,u')(\Delta G'^*(\iota))$$

Since F_0', F_1': $X \times I \to B$ are two liftings of the map pair

$$\begin{array}{ccc} X \times \dot{I} & \to & B \\ \downarrow & & \downarrow \\ X \times I & \to & b_0 \end{array}$$

it follows from corollary 5 that $d(F_0',F_1') = -\Delta G'^*(\iota)$, and by theorem 8.2.4, given $d \in H^{n-1}(X;\pi)$, there is a homotopy F_1': $X \times I \to B$ from $p \circ f_0$ to $p \circ f_1$ such that $\Delta G'^*(\iota) = (-1)^q\tau(d)$. Combining all of these, we see that $f_0 \simeq f_1$ if and only if there is $d \in H^{n-1}(X;\pi)$ such that

$$d(f_0,f_1) = \tau^{-1}\Delta(\theta,u')\tau(d) = S\Delta(\theta,u)(d) \quad \blacksquare$$

We summarize these results in the following classification theorem.

10 THEOREM *Let p: $E \to B$ be a principal fibration of type (G,q) over a space B of type (π,n) induced by a map $\bar\theta$: $B \to B'$ such that $\bar\theta^*(\iota') = \theta(\iota)$. Given a CW complex X, there is a map ψ: $[X;E] \to H^n(X;\pi)$ defined by $\psi[f] = (p \circ f)^*(\iota)$. Then $\operatorname{im}\psi = \{u \in H^n(X;\pi) \mid \theta(u) = 0\}$, and for every $u \in \operatorname{im}\psi$ the set $\psi^{-1}(u)$ is in one-to-one correspondence with*

$$H^{q-1}(X;G)/S\Delta(\theta,u)H^{n-1}(X;\pi)$$

PROOF We have already seen that $\operatorname{im}\psi$ is as described in the theorem. Given $u \in \operatorname{im}\psi$, let f_0: $X \to E$ be such that $\psi[f_0] = u$. Given any map

$f_1: X \to E$ such that $\psi[f_1] = u$, there is a map $f_1': X \to E$ homotopic to f_1 such that $p \circ f_1' = p \circ f_0$ (by the homotopy lifting property of p). To such a map f_1' we associate the element $d(f_0,f_1') \in H^{q-1}(X;G)$. In this way the set of maps $X \to E$ which are liftings of $p \circ f_0$ is mapped into $H^{q-1}(X;G)$, and by theorem 8.2.4, this map is surjective.

Two maps $f_1, f_2: X \to E$ such that $p \circ f_1 = p \circ f_0 = p \circ f_2$, are homotopic by lemma 9 if and only if $d(f_1,f_2) \in S\Delta(\theta,u)H^{n-1}(X;\pi)$. By lemma 8.2.3, $d(f_0,f_2) = d(f_0,f_1) + d(f_1,f_2)$, and so $f_1 \simeq f_2$ if and only if $d(f_0,f_1)$ and $d(f_0,f_2)$ belong to the same coset of $S\Delta(\theta,u)H^{n-1}(X;\pi)$ in $H^{q-1}(X;G)$. Hence the function which assigns the coset $d(f_0,f_1) + S\Delta(\theta,u)H^{n-1}(X;\pi)$ to a map $f_1: X \to E$ with $p \circ f_1 = p \circ f_0$ induces a bijection from $\psi^{-1}(u)$ to

$$H^{q-1}(X;G)/S\Delta(\theta,u)H^{n-1}(X;\pi) \quad \blacksquare$$

We now apply this to the complex projective space $P_m(\mathbf{C})$ for $m \geq 1$. There is a map $P_m(\mathbf{C}) \to P_\infty(\mathbf{C})$ and $P_\infty(\mathbf{C})$ is a space of type $(\mathbf{Z},2)$, by example 8.1.3. Furthermore, if ι is a characteristic element for $P_\infty(\mathbf{C})$ and B' is a space of type $(\mathbf{Z}, 2m + 2)$, there is a map $\bar\theta$: $P_\infty(\mathbf{C}) \to B'$ such that $\bar\theta^*(\iota') = (\iota)^{m+1}$. For the principal fibration $p: E \to P_\infty(\mathbf{C})$ induced by $\bar\theta$ there is a map $P_m(\mathbf{C}) \to E$ which is a $(2m + 2)$-equivalence. In this case the operation θ is the $(m + 1)$st-power operation, and therefore

$$\begin{aligned} S\Delta(\theta,u)(v) &= \tau^{-1}\Delta[j_1^* h^{*-1}(\tau(v)) + k^* u']^{m+1} \\ &= \tau^{-1}\Delta[(m + 1)k^*(u')^m \smile j_1^* h^{*-1}(\tau(v))] = (m + 1)u^m \smile v \end{aligned}$$

because $\tau(v) \smile \tau(v) = 0$. This gives the following application of theorem 10.

11 THEOREM *Let $\iota \in H^2(P_m(\mathbf{C});\mathbf{Z})$ be 2-characteristic for $P_m(\mathbf{C})$ and let X be a CW complex. Define ψ: $[X;P_m(\mathbf{C})] \to H^2(X;\mathbf{Z})$ by $\psi[f] = f^*(\iota)$. If* $\dim X \leq 2m + 2$, *then* $\operatorname{im}\psi = \{u \in H^2(X;\mathbf{Z}) \mid u^{m+1} = 0\}$. *If* $\dim X \leq 2m + 1$, *then ψ is surjective, and for a given $u \in H^2(X;\mathbf{Z})$, $\psi^{-1}(u)$ is in one-to-one correspondence with $H^{2m+1}(X;\mathbf{Z})/[(m + 1)u^m \smile H^1(X;\mathbf{Z})]$.* $\blacksquare$

5 THE SUSPENSION MAP

One of the most useful tools for the study of the homotopy groups of spaces is the suspension homomorphism from $\pi_q(X)$ to $\pi_{q+1}(SX)$. Iteration of this homomorphism yields a sequence of groups and homomorphisms

$$\pi_q(X) \to \pi_{q+1}(SX) \to \pi_{q+2}(S^2X) \to \cdots$$

This sequence has the stability property that from some point on, all the homomorphisms are isomorphisms. For a fixed X and q, therefore, there are only a finite number of different groups in the above sequence.

In this section we shall study the suspension map in some detail and establish the stability property. This will enable us to compute $\pi_{n+1}(S^n)$ for all n. Knowledge of these groups, combined with obstruction theory, will lead to the Steenrod classification theorem, which closes the section.[1]

We consider the category of pointed spaces and maps. There is a functorial suspension map $S\colon [X;Y] \to [SX;SY]$ such that $S[f] = [Sf]$. The exponential correspondence defines a natural isomorphism

$$[SX;SY] \approx [X;\Omega SY]$$

and we define $\bar{S}\colon [X;Y] \to [X;\Omega SY]$ to be the functorial map which is the composite of S with this isomorphism. The following result shows that $\bar{S}$ is induced by a map $Y \to \Omega SY$.

1 LEMMA *Let $\rho\colon Y \to \Omega SY$ be the map defined by $\rho(y)(t) = [y,t]$ for $y \in Y$ and $t \in I$. Then for any space X*

$$\bar{S} = \rho_\#\colon [X;Y] \to [X;\Omega SY]$$

PROOF The exponential correspondence takes the identity map $SY \subset SY$ to the map $\rho\colon Y \to \Omega SY$. Because of functorial properties of the exponential correspondence, it takes the composite

$$SX \xrightarrow{Sf} SY \subset SY$$

to the composite

$$X \xrightarrow{f} Y \xrightarrow{\rho} \Omega SY \quad \blacksquare$$

Thus, to study the suspension map S, we study the map ρ. To do this we use the fibration $PSY \to SY$, which has fiber ΩSY. With this in mind, let us investigate homology properties of fibrations over SY. We assume that $y_0 \in Y$ is a nondegenerate base point. We define $C_-Y = \{[y,t] \in SY \mid 0 \le t \le \frac{1}{2}\}$ and $C_+Y = \{[y,t] \in SY \mid \frac{1}{2} \le t \le 1\}$. Then $SY = C_-Y \cup C_+Y$, and there is a homeomorphism $Y \approx C_-Y \cap C_+Y$ (sending y to $[y,\frac{1}{2}]$) by means of which we identify Y with $C_-Y \cap C_+Y$. Let $S'Y$ be the unreduced suspension defined to be the quotient space of $Y \times I$ in which $Y \times 0$ is collapsed to one point and $Y \times 1$ is collapsed to another point and let C'_-Y, C'_+Y be analogous subspaces of $S'Y$ (so $C'_-Y \cap C'_+Y = Y$). The map collapsing $S'y_0$ in $S'Y$ is a collapsing map $k\colon S'Y \to SY$ such that $k(C'_-Y) = C_-Y$ and $k(C'_+Y) = C_+Y$.

2 LEMMA *If y_0 is a nondegenerate base point, the collapsing map $k\colon S'Y \to SY$ defines a homotopy equivalence from any pair consisting of the spaces $S'Y$, C'_-Y, C'_+Y, and Y to the corresponding pair consisting of SY, C_-Y, C_+Y, and Y.*

PROOF Because y_0 is a nondegenerate base point of Y, it follows, as in the proof of lemma 7.3.2*c*, that $Y \times \dot{I} \cup y_0 \times I \subset Y \times I$ is a cofibration. Let $[y,t]' \in S'Y$ denote the point of $S'Y$ determined by $(y,t) \in Y \times I$ under the quotient map $k'\colon Y \times I \to S'Y$. Let $H'\colon (Y \times \dot{I} \cup y_0 \times I) \times I \to S'Y$ be the homotopy defined by $H'(y,0,t) = [y_0,t/2]'$, $H'(y,1,t) = [y_0, (2-t)/2]'$, and $H'(y_0,t',t) = [y_0, (1-t)t' + t/2]'$. Then H' can be extended to a homotopy

[1] The first detailed study of the suspension map appears in H. Freudenthal, Über die Klassen der Sphärenabbildungen *I*, *Compositio Mathematica*, vol. 5, pp. 299–314, 1937.

$H''\colon Y \times I \times I \to S'Y$ such that $H''(y,t,0) = k'(y,t)$. Since $H''(y,0,t) = H''(y',0,t)$ and $H''(y,1,t) = H''(y',1,t)$ for all $y, y' \in Y$, it follows that there is a homotopy $H\colon S'Y \times I \to S'Y$ such that $H([y,t]', t') = H''(y,t,t')$. Then H is a homotopy from the identity map of $S'Y$ to a map which collapses $S'y_0$ to a single point such that $H(S'y_0 \times I) \subset S'y_0$. Since $H(B \times I) \subset B$ if $B = C'_-Y$, C'_+Y, or Y, the result follows from lemma 7.1.5. ∎

3 COROLLARY *If Y is a path-connected space with nondegenerate base point, then SY is simply connected.*

PROOF By lemma 2, $S'Y$ and SY have the same homotopy type, so it suffices to prove that $S'Y$ is simply connected. It is clearly path connected, being the quotient of the path-connected space $Y \times I$.

Let $U_- = \{[y,t]' \in S'Y \mid t < 1\}$ and $U_+ = \{[y,t]' \in S'Y \mid 0 < t\}$. Then U_- and U_+ are each open and contractible subsets of $S'Y$. If ω is any closed path in $S'Y$ at $[y_0,½]'$, there is a partition of I, say, $0 = t_0 < t_1 < \cdots < t_n = 1$, such that for each $1 \le i \le n$ either $\omega([t_{i-1},t_i]) \subset U_-$ or $\omega([t_{i-1},t_i]) \subset U_+$. Furthermore, it can be assumed that $\omega(t_i) \in U_- \cap U_+$ for all $0 \le i \le n$ (if some $\omega(t_i)$ is not in $U_- \cap U_+$, t_i can be omitted from the partition to obtain another partition of I satisfying the original hypothesis, and iteration of this procedure will lead to a partition having the additional property demanded). Since $U_- \cap U_+$ is homeomorphic to $Y \times \mathbf{R}$, it is path connected. For each i let ω_i be a path in $U_- \cap U_+$ from $\omega(t_{i-1})$ to $\omega(t_i)$ and let ω' be the closed path at $[y_0,½]'$ defined by $\omega'(t) = \omega_i((t - t_{i-1})/(t_i - t_{i-1}))$ for $t_{i-1} \le t \le t_i$. Because U_- and U_+ are each simply connected, $\omega \mid [t_{i-1},t_i]$ is homotopic to $\omega' \mid [t_{i-1},t_i]$ relative to $\{t_{i-1},t_i\}$. Therefore $\omega \simeq \omega'$ rel $\dot I$. Since ω' is a closed path in U_+, it is null homotopic. Therefore ω is null homotopic and $S'Y$ is simply connected. ∎

4 COROLLARY *Let Y have a nondegenerate base point and let $p\colon E \to SY$ be a fibration. Then $\{p^{-1}(C_-Y),p^{-1}(C_+Y)\}$ is an excisive couple in E.*

PROOF Let $p'\colon E' \to S'Y$ be the fibration induced from p by $k\colon S'Y \to SY$ and let $\bar k\colon E' \to E$ be the associated map. It follows from lemma 2 that $\bar k$ induces vertical isomorphisms in the commutative diagram

$$\begin{array}{ccc} H_*(p'^{-1}(C'_+Y),p'^{-1}(Y)) & \to & H_*(E',p'^{-1}(C'_-Y)) \\ \approx\downarrow & & \downarrow\approx \\ H_*(p^{-1}(C_+Y),p^{-1}(Y)) & \to & H_*(E,p^{-1}(C_-Y)) \end{array}$$

Since C'_+Y is a strong deformation retract of U_+ (with U_+ as defined in corollary 3) and Y is a strong deformation retract of $U_+ \cap C'_-Y$, it follows that $p'^{-1}(C'_+Y)$ and $p'^{-1}(Y)$ are strong deformation retracts of $p'^{-1}(U_+)$ and $p'^{-1}(U_+ \cap C'_-Y)$, respectively. This implies that $\{p'^{-1}(C'_-Y),p'^{-1}(C'_+Y)\}$ is an excisive couple. From the commutative diagram above, the result follows. ∎

Because C_+Y and C_-Y are contractible relative to y_0, it follows, as in Sec. 2.8, that for any fibration $p\colon E \to SY$ with fiber $F = p^{-1}(y_0)$ there are fiber homotopy equivalences $f_-: C_-Y \times F \to p^{-1}(C_-Y)$ and $g_+: p^{-1}(C_+Y) \to C_+Y \times F$ such that $f_- \mid y_0 \times F$ is homotopic to the map $(y_0,z) \to z$ and $g_+ \mid F$ is homotopic to the map $z \to (y_0,z)$. The corresponding *clutching function* $\mu: Y \times F \to F$ is defined by the equation

$$g_+f_-(y,z) = (y, \mu(y,z)) \qquad y \in Y, z \in F$$

Then $\mu \mid y_0 \times F$ is homotopic to the map $(y_0,z) \to z$.

5 THEOREM *Let $p: E \to SY$ be a fibration with $F = p^{-1}(y_0)$, where y_0 is a nondegenerate base point of Y. If $\mu: Y \times F \to F$ is a clutching function for p, there are exact sequences (any coefficient module)*

$$\cdots \to H_q(E) \to H_q(C_-Y \times F, Y \times F) \xrightarrow{\mu_*\partial} H_{q-1}(F) \xrightarrow{i_*} H_{q-1}(E) \to \cdots$$

$$\cdots \to H^q(E) \xrightarrow{i^*} H^q(F) \xrightarrow{\delta\mu^*} H^{q+1}(C_-Y \times F, Y \times F) \to H^{q+1}(E) \to \cdots$$

PROOF Consider the exact homology sequence of (E,F)

$$\cdots \to H_q(F) \xrightarrow{i_*} H_q(E) \to H_q(E,F) \xrightarrow{\partial} H_{q-1}(F) \to \cdots$$

Using homotopy properties and corollary 4, there are isomorphisms induced by inclusion maps

$$H_q(E,F) \xrightarrow{\approx} H_q(E,p^{-1}(C_+Y)) \xleftarrow{\approx} H_q(p^{-1}(C_-Y),p^{-1}(Y))$$

There is also a homotopy equivalence

$$f_-: (C_-Y \times F, Y \times F) \to (p^{-1}(C_-Y), p^{-1}(Y))$$

and a commutative diagram

$$\begin{tikzcd}
H_q(E,F) \arrow[r, "\approx"'] \arrow[d, "\partial"'] & H_q(E,p^{-1}(C_+Y)) \arrow[d, "\partial"'] & H_q(p^{-1}(C_-Y),p^{-1}(Y)) \arrow[l, "\approx"'] \arrow[d, "\partial"'] & H_q((C_-Y,Y) \times F) \arrow[l, "f_{-*}", "\approx"'] \arrow[d, "\partial"'] \\
H_{q-1}(F) \arrow[r, "j_*", "\approx"'] & H_{q-1}(p^{-1}(C_+Y)) & H_{q-1}(p^{-1}(Y)) \arrow[l] & H_{q-1}(Y \times F) \arrow[l, "(f_- \mid Y \times F)_*", "\approx"']
\end{tikzcd}$$

There is also a homotopy equivalence $g_+: p^{-1}(C_+Y) \to C_+Y \times F$ and isomorphisms

$$H_{q-1}(p^{-1}(C_+Y)) \xrightarrow[\approx]{g_{+*}} H_{q-1}(C_+Y \times F) \xrightarrow{\approx} H_{q-1}(F)$$

where the right-hand homomorphism is induced by projection to the second factor. Because $g_+ \mid F$ is homotopic to the map $z \to (y_0,z)$, the above composite equals $j_*{}^{-1}$. By definition, μ is the composite

$$Y \times F \xrightarrow{f_- \mid Y \times F} p^{-1}(Y) \subset p^{-1}(C_+Y) \xrightarrow{g_+} C_+Y \times F \to F$$

Therefore there is a commutative diagram

$$\begin{tikzcd}
H_q(E,F) \arrow[r, "\approx"'] \arrow[d, "\partial"'] & H_q((C_-Y,Y) \times F) \arrow[d, "\partial"] \\
H_{q-1}(F) & H_{q-1}(Y\times F) \arrow[l, "\mu_*"']
\end{tikzcd}$$

The desired exact sequence for homology follows on replacing $H_q(E,F)$ by $H_q((C_-Y,Y) \times F)$ and ∂ by $\mu_* \partial$ in the homology sequence of (E,F). A similar argument establishes the exactness of the cohomology sequence. ■

Specializing to the case where $Y = S^{n-1}$, by lemma 1.6.6, $S(S^{n-1})$ is homeomorphic to S^n, and we obtain the following exact *Wang sequence* of a fibration over S^n.

6 COROLLARY *Let $p: E \to S^n$ be a fibration with fiber F. There are exact sequences*

$$\cdots \to H_q(F) \xrightarrow{i_*} H_q(E) \to H_{q-n}(F) \to H_{q-1}(F) \to \cdots$$

$$\cdots \to H^q(E) \xrightarrow{i^*} H^q(F) \xrightarrow{\theta} H^{q-n+1}(F) \to H^{q+1}(E) \to \cdots$$

If the second sequence has coefficients in a commutative ring with a unit, then

$$\theta(u \smile v) = \theta(u) \smile v + (-1)^{(n-1)\deg u} u \smile \theta(v)$$

PROOF Letting $Y = S^{n-1}$ in theorem 5, we have (C_-Y,Y) homeomorphic to (E^n,S^{n-1}). Therefore

$$H_q((C_-Y,Y) \times F) \approx H_q((E^n,S^{n-1}) \times F) \approx H_{q-n}(F)$$

and the exact sequences result from the exact sequences of theorem 5 on replacing $H_q(C_-Y \times F,\ Y \times F)$ and $H^q(C_-Y \times F,\ Y \times F)$ by $H_{q-n}(F)$ and $H^{q-n}(F)$, respectively. The additional fact concerning θ results from the observation that for the map $\mu^*: H^q(F) \to H^q(S^{n-1} \times F)$ the definitions are such that

$$\mu^*(u) = 1 \times u + s^* \times \theta(u)$$

where $s^* \in H^{n-1}(S^{n-1})$ is a suitable generator. Then, since $s^* \smile s^* = 0$,

$$\begin{aligned} 1 \times (u \smile v) + s^* \times \theta(u \smile v) &= \mu^*(u \smile v) \\ &= [1 \times u + s^* \times \theta(u)] \smile [1 \times v + s^* \times \theta(v)] \\ &= 1 \times (u \smile v) + s^* \times [\theta(u) \smile v + (-1)^{(n-1)\deg u} u \smile \theta(v)] \end{aligned}$$

This implies the multiplicative property of θ. ■

We now specialize to the path fibration $p: PSY \to SY$ with fiber ΩSY. In this case there is the following simple expression for a clutching function.

7 LEMMA *Let $s_-: C_-Y \to p^{-1}(C_-Y)$ and $s_+: C_+Y \to p^{-1}(C_+Y)$ be sections of the fibration $p: PSY \to SY$ such that $s_-(y_0)$ and $s_+(y_0)$ are both null homotopic loops. Then the map $\mu: Y \times \Omega SY \to \Omega SY$ defined by*

$$\mu(y,\omega) = (\omega * s_-(y)) * s_+(y)^{-1}$$

is a clutching function for p.

PROOF Such sections exist because C_-Y and C_+Y are contractible relative to y_0. We define fiber-preserving maps

$$f_-\colon C_-Y \times \Omega SY \to p^{-1}(C_-Y) \qquad g_-\colon p^{-1}(C_-Y) \to C_-Y \times \Omega SY$$
$$f_+\colon C_+Y \times \Omega SY \to p^{-1}(C_+Y) \qquad g_+\colon p^{-1}(C_+Y) \to C_+Y \times \Omega SY$$

by $f_-(z,\omega) = \omega * s_-(z)$ and $g_-(\omega) = (p(\omega), \omega * (s_-p(\omega))^{-1})$ and $f_+(z,\omega) = \omega * s_+(z)$ and $g_+(\omega) = (p(\omega), \omega * (s_+p(\omega))^{-1})$, respectively. It is easy to verify that $g_- \circ f_-$ is fiber homotopic to the identity map of $C_-Y \times \Omega SY$ and $f_- \circ g_-$ is fiber homotopic to the identity map of $p^{-1}(C_-Y)$. Therefore f_- is a fiber homotopy equivalence. Similarly, g_+ is a fiber homotopy equivalence. Furthermore, $f_-(y_0,\omega) = \omega * s_-(y_0)$ is homotopic to the map $(y_0,\omega) \to \omega$ because $s_-(y_0)$ is null homotopic. Similarly, for $\omega \in \Omega SY$, $g_+(\omega) = (y_0, \omega * s_+(y_0)^{-1})$ is homotopic to the map $\omega \to (y_0,\omega)$. Therefore the composite

$$Y \times \Omega SY \xrightarrow{f_-} p^{-1}(Y) \xrightarrow{g_+} Y \times \Omega SY \to \Omega SY$$

is a clutching function for p. This composite is the map

$$(y,\omega) \to (\omega * s_-(y)) * s_+(y)^{-1} \quad \blacksquare$$

Let s_- and s_+ be sections as in lemma 7 and let $\mu'\colon Y \to \Omega SY$ be defined by $\mu'(y) = s_-(y) * s_+(y)^{-1}$. μ' is called a *characteristic map* for the fibration $p\colon PSY \to SY$.

8 COROLLARY *Let $\mu'\colon Y \to \Omega SY$ be a characteristic map for the fibration $p\colon PSY \to SY$. The map $Y \times \Omega SY \to \Omega SY$ sending (y,ω) to $\omega * \mu'(y)$ is homotopic to a clutching function for p.*

PROOF This follows from lemma 7, because the map

$$(y,\omega) \to (\omega * s_-(y)) * s_+(y)^{-1}$$

is clearly homotopic to the map $(y,\omega) \to \omega * (s_-(y) * s_+(y)^{-1}) = \omega * \mu'(y)$. $\blacksquare$

The following theorem is the main part of the proof of the suspension theorem.

9 THEOREM *Let Y be n-connected for some $n \geq 0$ and let y_0 be a nondegenerate base point of Y. If $\mu'\colon Y \to \Omega SY$ is a characteristic map for the fibration $p\colon PSY \to SY$, then μ' induces an isomorphism*

$$\mu'_*\colon H_q(Y) \approx H_q(\Omega SY) \qquad q \leq 2n + 1$$

PROOF By corollary 3, SY is simply connected. By corollary 4, $\{C_-Y,C_+Y\}$ is an excisive couple, and from the exactness of the reduced Mayer-Vietoris sequence, $\tilde{H}_q(SY) \approx \tilde{H}_{q-1}(Y)$. Combining these with the absolute Hurewicz isomorphism theorem, SY is $(n + 1)$-connected. Therefore ΩSY is n-connected. Because PSY is contractible, it follows from the version of theorem 5, using reduced modules, that there is an isomorphism

$$\mu_*\partial\colon H_q((C_-Y,Y) \times \Omega SY) \approx \tilde{H}_{q-1}(\Omega SY)$$

If ω_0 is the constant loop, then because ΩSY is n-connected and (C_-Y,Y) is $(n + 1)$-connected, it follows from the Künneth theorem that the inclusion map $(C_-Y,Y) \times \omega_0 \subset (C_-Y,Y) \times \Omega SY$ induces an isomorphism

$$H_q((C_-Y,Y) \times \omega_0) \approx H_q((C_-Y,Y) \times \Omega SY) \qquad q \leq 2n + 2$$

Let μ: $Y \times \Omega SY \to \Omega SY$ be a clutching function which is homotopic to the map $(y,\omega) \to \omega * \mu'(y)$ (such a μ exists, by corollary 8). Since $\mu(y,\omega_0)$ is homotopic to the map $y \to \mu'(y)$, there is a commutative diagram

$$\begin{tikzcd}
H_q(C_-Y,Y) \arrow[r, "\approx"'] \arrow[d, "\partial"] & H_q((C_-Y,Y) \times \omega_0) \arrow[r] \arrow[d, "\partial"] & H_q((C_-Y,Y) \times \Omega SY) \arrow[d, "\partial"] \\
\tilde{H}_{q-1}(Y) \arrow[r, "\approx"'] \arrow[dr, "\mu'_*"'] & \tilde{H}_{q-1}(Y \times \omega_0) \arrow[r] & \tilde{H}_{q-1}(Y \times \Omega SY) \arrow[dl, "\mu_*"] \\
& \tilde{H}_{q-1}(\Omega SY) &
\end{tikzcd}$$

The result follows from the commutativity of this diagram. ∎

10 COROLLARY *Let Y have a nondegenerate base point. If Y is n-connected for $n \geq 0$, the map ρ: $Y \to \Omega SY$ induces an isomorphism*

$$\rho_* : H_q(Y) \approx H_q(\Omega SY) \qquad q \leq 2n + 1$$

PROOF Let s_-: $C_-Y \to p^{-1}(C_-Y)$ and s_+: $C_+Y \to p^{-1}(C_+Y)$ be the sections defined by $s_-[y,t](t') = [y,tt']$ and $s_+[y,t](t') = [y, 1 - t' + tt']$. The corresponding characteristic map is equal to the map ρ: $Y \to \Omega SY$. The result follows from theorem 9. ∎

We are now ready for the following *suspension theorem.*[1]

11 THEOREM *Let Y be n-connected for $n \geq 1$ with a nondegenerate base point and let X be a pointed CW complex. Then the suspension map*

$$S: [X;Y] \to [SX;SY]$$

is surjective if $\dim X \leq 2n + 1$ *and bijective if* $\dim X \leq 2n$.

PROOF Because Y and ΩSY are simply connected, it follows from corollary 10 and the Whitehead theorem that ρ is a $(2n + 1)$-equivalence. The result follows from corollary 7.6.23 and lemma 1. ∎

Let Y be a space with a nondegenerate base point. Then SY also has a nondegenerate base point and is path connected, S^2Y is simply connected, and S^mY is $(m - 1)$-connected. If X is a CW complex, so is S^mX, and $\dim (S^mX) = m + \dim X$. Hence, if X is finite dimensional and $m \geq 2 + \dim X$, it follows from theorem 11 that S: $[S^mX; S^mY] \approx [S^{m+1}X; S^{m+1}Y]$. Therefore, for any finite-dimensional CW complex X the sequence

$$[X;Y] \xrightarrow{S} [SX;SY] \xrightarrow{S} \cdots \xrightarrow{S} [S^mX;S^mY] \xrightarrow{S} \cdots$$

[1] For a general relative form of this theorem see E. Spanier and J. H. C. Whitehead, The theory of carriers and S-theory, in "Algebraic Geometry and Topology" (a symposium in honor of S. Lefschetz), Princeton University Press, Princeton, N.J., 1957, pp. 330–360.

consists of isomorphisms from some point on. Taking $X = S^{n+k}$ and $Y = S^n$ and recalling that the suspension of a sphere is a sphere, we see that there is a sequence

$$\pi_{n+k}(S^n) \xrightarrow{S} \pi_{n+k+1}(S^{n+1}) \xrightarrow{S} \cdots$$

consisting of isomorphisms from some point on. The direct limit of this sequence is called the *k-stem*. It follows from theorem 11 that the k-stem is isomorphic to $\pi_{2k+2}(S^{k+2})$. In particular, the 0-stem is infinite cyclic. The following result determines the 1-stem.

12 THEOREM $\pi_4(S^3) \approx \mathbf{Z}_2$.

PROOF Let $u_0 \in H^0(\Omega S^3)$ be the unit integral class and define generators $u_i \in H^{2i}(\Omega S^3)$, by induction on i from the exactness of the Wang sequence in corollary 6 for the fibration $PS^3 \to S^3$, by the equation

$$\theta(u_{i+1}) = u_i \qquad i \geq 0$$

Because θ is a derivation, $\theta(u_1 \smile u_1) = 2u_1$, whence $u_1 \smile u_1 = 2u_2$. We know $\pi_2(\Omega S^3) \approx \pi_3(S^3)$ is infinite cyclic. It follows that ΩS^3 can be imbedded in a space X of type $(\mathbf{Z},2)$ such that the inclusion map $\Omega S^3 \subset X$ induces an isomorphism $\pi_2(\Omega S^3) \approx \pi_2(X)$. Since $P_\infty(\mathbf{C})$ is also a space of type $(\mathbf{Z},2)$, it follows that $H^*(X) \approx H^*(P_\infty(\mathbf{C})) \approx \lim_{\leftarrow} \{H^*(P_j(\mathbf{C}))\}$ is a polynomial algebra with a single generator $v \in H^2(X)$, and v can be chosen so that $v \mid \Omega S^3 = u_1$.

An easy computation using the exact cohomology sequence of $(X,\Omega S^3)$ establishes that $H^q(X,\Omega S^3) = 0$ for $q < 5$ and $H^5(X,\Omega S^3) \approx \mathbf{Z}_2$. By the universal-coefficient formula, $H_q(X,\Omega S^3) = 0$ for $q < 4$ and $H_4(X,\Omega S^3) \approx \mathbf{Z}_2$. By the relative Hurewicz isomorphism theorem, $\pi_4(X,\Omega S^3) \approx \mathbf{Z}_2$. Because $\pi_3(X) = 0 = \pi_4(X)$, we have $\pi_4(X,\Omega S^3) \overset{\partial}{\approx} \pi_3(\Omega S^3) \approx \pi_4(S^3)$. ∎

The $(n-2)$-fold suspension of a generator of $\pi_3(S^2)$ is a generator of $\pi_{n+1}(S^n)$ (because S: $\pi_3(S^2) \to \pi_4(S^3)$ is an epimorphism, by theorem 11). Attaching a cell to S^n by this map must, therefore, kill $\pi_{n+1}(S^n)$. The resulting CW complex has the same homotopy type as the $(n-2)$-fold suspension of the complex projective plane $P_2(\mathbf{C})$. Therefore we have proved the following result.

13 COROLLARY $\pi_{n+1}(S^{n-2}(P_2(\mathbf{C}))) = 0 \qquad n \geq 2$ ∎

We want to classify maps of an $(n+1)$-complex into S^n. For $n = 2$ this is given by the case $m = 1$ of theorem 8.4.11. By using the standard Postnikov factorization of S^n, we are reduced to classifying maps of an $(n+1)$-complex into E, where $p: E \to B$ is a principal fibration of type $(\mathbf{Z}_2, n+2)$, with base space B a space of type $(\mathbf{Z},n)$. This fibration determines a cohomology operation θ_n of type $(n, n+2; \mathbf{Z},\mathbf{Z}_2)$.

14 LEMMA *For $n > 2$ the cohomology operation θ_n is $Sq^2 \circ \mu_*$, where $\mu_*: H^n(X;\mathbf{Z}) \to H^n(X;\mathbf{Z}_2)$ is induced by the coefficient homomorphism $\mu: \mathbf{Z} \to \mathbf{Z}_2$.*

PROOF $S^n \subset S^{n-2}(P_2(\mathbf{C}))$ is not a retract, by theorem 12 and corollary 13. Therefore $\theta_n\colon H^n(S^{n-2}(P_2(\mathbf{C}));\mathbf{Z}) \to H^{n+2}(S^{n-2}(P_2(\mathbf{C}));\mathbf{Z}_2)$ is nontrivial (if θ_n were trivial, there would be a map $f\colon S^{n-2}(P_2(\mathbf{C})) \to S^n$ such that

$$f^*\colon H^n(S^n;\mathbf{Z}) \approx H^n(S^{n-2}(P_2(\mathbf{C}));\mathbf{Z})$$

is inverse to the restriction map $H^n(S^{n-2}(P_2(\mathbf{C}));\mathbf{Z}) \approx H^n(S^n;\mathbf{Z})$, and such a map f would be homotopic to a weak retraction). Since $Sq^2 \circ \mu_*$ is also nontrivial, it follows that $\theta_n = Sq^2 \circ \mu_*$ in the space $S^{n-2}(P_2(\mathbf{C}))$.

The rest of the argument follows by showing that $S^{n-2}(P_2(\mathbf{C}))$ is universal for θ_n and $Sq^2 \circ \mu_*$. Let X be any CW complex of dimension $\leq n + 2$ and let $u \in H^n(X;\mathbf{Z})$. Because $\pi_{n+1}(S^{n-2}(P_2(\mathbf{C}))) = 0$, there is a map $f\colon X \to S^{n-2}(P_2(\mathbf{C}))$ such that $f^*v = u$, where v is a generator of $H^n(S^{n-2}(P_2(\mathbf{C})))$. By the naturality of θ_n and $Sq^2 \circ \mu_*$, it follows that

$$\theta_n(u) = \theta_n f^* v = f^*\theta_n v = f^* Sq^2\mu_* v = Sq^2\mu_*(u)$$

Since this is true for every CW complex of dimension $\leq n + 2$ and θ_n and $Sq^2 \circ \mu_*$ are operations of type $(n, n + 2; \mathbf{Z},\mathbf{Z}_2)$, it is true for every CW complex. ∎

Combining lemma 14 with theorem 8.4.10 yields the following *Steenrod classification theorem.*[1]

15 THEOREM *Let $s^* \in H^n(S^n;\mathbf{Z})$ be a generator, where $n > 2$, and let X be a CW complex. Then the map $\psi\colon [X;S^n] \to H^n(X;\mathbf{Z})$ has image equal to $\{u \in H^n(X;\mathbf{Z}) \mid Sq^2\mu_*(u) = 0\}$ if* dim $X \leq n + 2$, *and if* dim $X \leq n + 1$, *$\psi^{-1}(u)$ is in one-to-one correspondence with $H^{n+1}(X;\mathbf{Z}_2)/Sq^2\mu_* H^{n-1}(X;\mathbf{Z})$.* ∎

EXERCISES

A SPACES OF TYPE (π,n)

1 For p an integer let $L_n(p)$ be the generalized lens space $L_n(p) = L(p, \overbrace{1, \ldots, 1}^{n})$. Show that $L_n(p) \subset L_{n+1}(p)$ and that $L_\infty(p) = \bigcup_n L_n(p)$ topologized with the topology coherent with $\{L_n(p)\}$ is a space of type $(\mathbf{Z}_p,1)$.

2 If X is a CW complex of type (π,n) for $n > 1$ and Y is a CW complex, prove that

$$\pi_n(X \vee Y) \approx \pi_n(Y) \oplus \bigoplus_{\lambda \in \pi_1(Y)} \pi_\lambda$$

where $\pi_\lambda = \pi$ for each $\lambda \in \pi_1(Y)$.

3 Given a sequence of groups $\{\pi_q\}_{q\geq 1}$, with π_q abelian for $q > 1$, and given an action of π_1 as a group of operators on π_q for $q > 1$, prove that there is a space Y which realizes this sequence (that is, $\pi_q(Y) \approx \pi_q$ and $\pi_1(Y)$ acting on $\pi_q(Y)$ corresponds to the action of π_1 on π_q).

[1] See N. E. Steenrod, Products of cocycles and extensions of mappings, *Annals of Mathematics,* vol. 48, pp. 290–320, 1947.

B EXACT SEQUENCES CONTAINING $g_\#$

Let g: $(Y,B) \to (Y',B')$ be a base-point-preserving map and let $g' = g \mid Y$: $Y \to Y'$ and $g'' = g \mid B$: $B \to B'$.

1 Prove that $E_{g''}$ is a subspace of $E_{g'}$ and $p_{g''} = p_{g'} \mid E_{g''}$.

2 Define p: $(E_{g'},E_{g''}) \to (Y,B)$ so that $p \mid E_{g'} = p_{g'}$ and j: $(\Omega Y',\Omega B') \to (E_{g'},E_{g''})$ so that $j(\omega) = (y_0,\omega)$. Prove that there is an exact sequence

$$(\Omega Y,\Omega B) \xrightarrow{\Omega g} (\Omega Y',\Omega B') \xrightarrow{j} (E_{g'},E_{g''}) \xrightarrow{p} (Y,B) \xrightarrow{g} (Y',B')$$

3 Prove that there is an exact sequence

$$\cdots \xrightarrow{\Omega^n j} \Omega^n(E_{g'},E_{g''}) \xrightarrow{\Omega^n p} \Omega^n(Y,B) \xrightarrow{\Omega^n g} \Omega^n(Y',B') \to \cdots \xrightarrow{g} (Y',B')$$

4 Define a map $(\Omega Y' \times E_{g'}, \Omega B' \times E_{g''}) \to (E_{g'},E_{g''})$ sending $\omega \times (y_0,\omega')$ to $(y_0, \omega * \omega')$ and use this to define an action $a \top b$ of $[X,A; \Omega Y',\Omega B']$ on the left on $[X,A; E_{g'},E_{g''}]$. Prove that $p_\#(b_1) = p_\#(b_2)$ for b_1, $b_2 \in [X,A; E_{g'},E_{g''}]$ if and only if there is $a \in [X,A; \Omega Y',\Omega B']$ such that $b_1 = a \top b_2$.

5 Prove that $j_\#(a_1) = j_\#(a_2)$ for a_1, $a_2 \in [X,A; \Omega Y',\Omega B']$ if and only if there is $c \in [X,A; \Omega Y,\Omega B]$ such that $a_1 = a_2(\Omega g)_\#(c)$.

C EXAMPLES

1 Find an example of an n-dimensional polyhedron X, with $n > 1$, and a map f: $X \to S^n$ such that $f_*: \tilde{H}_*(X) \to \tilde{H}_*(S^n)$ is trivial but f is not homotopic to a constant map.

2 Let X be an n-dimensional polyhedron. Prove that f, g: $X \to S^n$ are homotopic if and only if $f_* = g_*: H_n(X;G) \to H_n(S^n;G)$ for $G = \mathbf{Z}_p$ with p a prime, and for $G = \mathbf{R}$.

3 Compute the cohomotopy group $\pi^{2m-1}(P_m(\mathbf{C}))$ for $m \geq 2$.

4 Let (Y,B) be a pair which is $(n-1)$-connected for $n \geq 2$, with a simple inclusion map $B \subset Y$, and let $\iota \in H^n(Y,B; \pi)$ be n-characteristic for (Y,B). If (X,A) is a relative CW complex and f: $(X,A) \to (Y,B)$, prove that $f^*(\iota) \in H^n(X,A; \pi)$ is the first obstruction to deforming f relative to A to a map from X to B.

D SUSPENSION

1 Let X be an $(n-1)$-connected CW complex of dimension $\leq 2n-1$. Prove that there is a CW complex Y such that SY has the same homotopy type as X. [*Hint:* Show that X has the same homotopy type as a CW complex X', with $(X')^{n-1}$ a single point. Construct Y inductively by desuspending the attaching maps of the cells of X'.]

2 Let A and B be closed subsets of a space X such that $X = A \cup B$. Assume that f, g: $X \to Y$ are such that $f(A) = y_0 = g(B)$ and define h: $X \to Y$ so that $h \mid A = g \mid A$ and $h \mid B = f \mid B$. Prove that, in $[SX;SY]$,

$$[Sf][Sg] = [Sh]$$

3 Let X and Y be path-connected pointed CW complexes. Prove that a map f: $X \to Y$ has the property that $S^k f$: $S^k X \to S^k Y$ is a homotopy equivalence for some $k \geq 0$ if and only if Sf: $SX \to SY$ is a homotopy equivalence. [*Hint:* Show that either condition is equivalent to the condition $f_*: H_*(X) \approx H_*(Y)$.]

4 Let X and Y be path-connected pointed CW complexes and let p_1: $X \times Y \to X$ and p_2: $X \times Y \to Y$ be the projections and k: $X \times Y \to X \# Y = X \times Y / X \vee Y$ the collapsing map. Regard all three as maps into $X \vee Y \vee (X \# Y)$ and prove that

$$((Sp_1) * (Sp_2)) * (Sk)\colon S(X \times Y) \to S(X \vee Y \vee X \# Y)$$

is a homotopy equivalence.

5 Show that there exist CW complexes with different homotopy type whose suspensions have the same homotopy type.

E THE SUSPENSION CATEGORY

Let $\{X,A;\ Y,B\} = \lim_{\rightarrow} [S^kX,S^kA;\ S^kY,S^kB]$, and for q an integer (positive or negative) let $\{X,A;\ Y,B\}_q = \lim_{\rightarrow} [S^{k+q}X,S^{k+q}A;\ S^kY,S^kB]$. If $\alpha\colon S^{k+q}(X,A) \to S^k(Y,B)$, then $\{\alpha\}$ will denote the corresponding element of $\{X,A;\ Y,B\}_q$.

1 Prove that there is a pairing

$$\{Y,B;\ Z,C\}_p \otimes \{X,A;\ Y,B\}_q \to \{X,A;\ Z,C\}_{p+q}$$

sending $\{\alpha\} \otimes \{\beta\}$ to $\{\alpha \circ \beta\}$, where

$$S^{p+q+k}(X,A) \xrightarrow{\beta} S^{p+k}(Y,B) \xrightarrow{\alpha} S^k(Z,C)$$

2 If A is closed in X and (X,A) has a nondegenerate base point, prove that $\{(C_-X,C_-A),\ (C_+X,C_+A)\}$ is an excisive couple of subsets. Let S: $H_q(X,A) \approx H_{q+1}(SX,SA)$ and S: $H^q(X,A) \approx H^{q+1}(SX,SA)$ be the isomorphisms of the corresponding relative Mayer-Vietoris sequences.

3 Prove that there are pairings

$$\{X,A;\ Y,B\}_p \otimes H_q(X,A) \to H_{p+q}(Y,B)$$
$$\{X,A;\ Y,B\}_p \otimes H^r(Y,B) \to H^{r-p}(X,A)$$

sending $\{\alpha\} \otimes z$ to $S^{-k}(\alpha_*(S^{k+p}z))$ and $\{\alpha\} \otimes u$ to $S^{-k-p}(\alpha^*(S^ku))$ for $z \in H_q(X,A)$, $u \in H^r(Y,B)$, and $\alpha\colon S^{k+p}(X,A) \to S^k(Y,B)$.

4 If (X,A) is a pointed pair, with $A \subset X$ a cofibration, and Y is a pointed space, prove that there is an exact sequence

$$\cdots \to \{X;Y\}_q \to \{A;Y\}_q \to \{X/A;Y\}_{q-1} \to \{X;Y\}_{q-1} \to \cdots$$

5 Let X be a pointed space and (Y,B) a pointed pair, with $B \subset Y$ a cofibration. If $f\colon X \to Y$ is such that the composite $X \xrightarrow{f} Y \xrightarrow{k} Y/B$ is null homotopic, prove that Sf is homotopic to the composite $SX \xrightarrow{f'} SB \subset SY$ for some f'. Deduce the existence of an exact sequence

$$\cdots \to \{X;B\}_q \to \{X;Y\}_q \to \{X;Y/B\}_q \to \{X;B\}_{q-1} \to \cdots$$

F DUALITY IN THE SUSPENSION CATEGORY[1]

In this group of exercises all spaces are assumed to be finite CW complexes with base points. An *n-duality* is an element $u \in \{X^* \# X;\ S^0\}_{-n}$ such that the map sending $\{\alpha\} \in \{S^0;X^*\}_q \approx \{S^q;X^*\}$ to $u \circ (\{\alpha\} \# \{1_X\}) \in \{S^q \# X;\ S^0\}_{-n} \approx \{X;S^0\}_{q-n}$ is an isomorphism

$$D_u\colon \{S^0;X^*\}_q \approx \{X;S^0\}_{q-n}$$

and the map sending $\{\beta\} \in \{S^0;X\}_q \approx \{S^q;X\}$ to $u \circ (\{1_{X^*}\} \# \{\beta\}) \in \{X^* \# S^q;\ S^0\}_{-n} \approx \{X^*;S^0\}_{q-n}$ is an isomorphism

[1] See E. Spanier, Function spaces and duality, *Annals of Mathematics*, vol. 70, pp. 338–378, 1959, for a different development of this topic. The one given in the text is based on a suggestion of P. Freyd and has also been considered by D. Husemoller.

$$D^u\colon \{S^0;X\}_q \approx \{X^*;S^0\}_{q-n}$$

1 If $f\colon S^p \# S^q \to S^{p+q}$ is a homeomorphism, prove that $\{f\} \in \{S^p \# S^q;\, S^0\}_{-p-q}$ is a $(p+q)$-duality.

2 If $u \in \{X^* \# X;\, S^0\}_{-n}$ is an n-duality, prove that the element $u' \in \{X \# X^*;\, S^0\}_{-n}$ corresponding to u under the homeomorphism $X \# X^* \to X^* \# X$ is also an n-duality.

3 If $u \in \{X^* \# X;\, S^0\}_{-n}$ is an n-duality, prove that for any Y and Z there are isomorphisms

$$D_u\colon \{Y;\, Z \# X^*\}_q \approx \{Y \# X;\, Z\}_{q-n}$$
$$D^u\colon \{Y;\, X \# Z\}_q \approx \{X^* \# Y;\, Z\}_{q-n}$$

such that $D_u\{\alpha\} = (\{1_Z\} \# u) \circ (\{\alpha\} \# \{1_X\})$ for $\{\alpha\} \in \{Y;\, Z \# X^*\}_q$ and $D^u\{\beta\} = (u \# \{1_Z\}) \circ (\{1_{X^*}\} \# \{\beta\})$ for $\{\beta\} \in \{Y;\, X \# Z\}_q$. (*Hint:* If Y and Z are spheres, this is true by definition of n-duality. For arbitrary Y and Z use induction on the number of cells and the five lemma.)

Given n-dualities $u \in \{X^* \# X;\, S^0\}_{-n}$ and $v \in \{Y^* \# Y;\, S^0)_{-n}$, define an isomorphism

$$D(u,v)\colon \{X;Y\}_q \approx \{Y^*;X^*\}_q$$

so that the following diagram is commutative:

$$\begin{tikzcd} \{X;Y\}_q \arrow[rr, "D(u,v)"] \arrow[dr, "D^v"', "\approx"] & & \{Y^*;X^*\}_q \arrow[dl, "D_u", "\approx"'] \\ & \{Y^* \# X;\, S^0\}_{q-n} & \end{tikzcd}$$

4 Prove that $D(v',u') = (D(u,v))^{-1}\colon \{Y^*;X^*\}_q \approx \{X;Y\}_q$.

5 If $u \in \{X^* \# X;\, S^0\}_{-n}$, $v \in \{Y^* \# Y;\, S^0\}_{-n}$, and $w \in \{Z^* \# Z;\, S^0\}_{-n}$ are n-dualities and $\{\alpha\} \in \{X;Y\}_p$ and $\{\beta\} \in \{Y;Z\}_q$, prove that, in $\{Z^*;X^*\}_{p+q}$,

$$D(u,w)(\{\beta\} \circ \{\alpha\}) = (D(u,v)\{\alpha\}) \circ (D(v,w)\{\beta\})$$

Assume that $f\colon X^* \# X \to S^n$ and $g\colon Y^* \# Y \to S^n$ are such that $\{f\}$ and $\{g\}$ are n-dualities and let $\alpha\colon X \to Y$ and $\beta\colon Y^* \to X^*$ be maps such that

$$f \circ (\beta \# 1_X) \simeq g \circ (1_{Y^*} \# \alpha)\colon Y^* \# X \to S^n$$

[which implies $D(\{f\},\{g\})\{\alpha\} = \{\beta\}$]. Let C_α and C_β be the mapping cones of α and β, respectively, and consider the coexact sequences

$$X \xrightarrow{\alpha} Y \xrightarrow{i} C_\alpha \xrightarrow{k} SX \xrightarrow{S\alpha} SY$$
$$Y^* \xrightarrow{\beta} X^* \xrightarrow{i'} C_\beta \xrightarrow{k'} SY^* \xrightarrow{S\beta} SX^*$$

6 Prove that there is a map $h\colon C_\beta \# C_\alpha \to S^{n+1}$ such that the following squares are homotopy commutative:

$$\begin{tikzcd} X^* \# C_\alpha \arrow[r, "1 \# k"] \arrow[d, "i' \# 1"'] & X^* \# SX \leftrightarrow S(X^* \# X) \arrow[d, "Sf"] \\ C_\beta \# C_\alpha \arrow[r, "h"] & S^{n+1} \end{tikzcd} \qquad \begin{tikzcd} C_\beta \# Y \arrow[r, "1 \# i"] \arrow[d, "k' \# 1"'] & C_\beta \# C_\alpha \arrow[d, "h"] \\ SY^* \# Y \leftrightarrow S(Y^* \# Y) \arrow[r, "Sg"] & S^{n+1} \end{tikzcd}$$

Deduce that $\{h\} \in \{C_\beta \# C_\alpha;\, S^0\}_{-n-1}$ is an $(n+1)$-duality.

7 For any X there is an integer n for which there exists a space X^* and an n-duality $u \in \{X^* \# X;\, S^0\}_{-n}$. (*Hint:* Prove this by induction on the number of cells of X, using exercises 1 and 6 above.)

CHAPTER NINE

SPECTRAL SEQUENCES AND HOMOTOPY GROUPS OF SPHERES